David Cummings

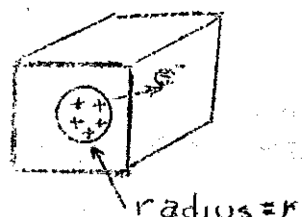
Space Science 500 ~~SOACE SCUEBCE 500~~ 9/18/63

Lecture I

This morning we are going to start out to define a plasma...that is, what it's properties are, what it means, and when you can use plasma concepts, and when you get to single particle concepts.

First of all, a plasma is essentially electrically neutral. A plasma is a collection of charged particles. The deviation from neutrality in any point of plasma cannot be much greater than the amount dictated by thermal energy.

Let's say we have a volume, a block of plasma that contains electrons and protons, and we consider a very small sphere:



in which, by a statistical fluctuation, all the electrons have gone out and left only protons. The question is, "How far can the electron move away before it gets brought back by the excess positive charge in this sphere?"

The potential at the surface of the sphere:

$$\phi = \frac{q}{4\pi\epsilon_0 r} \qquad q = ne\left(\frac{4\pi}{3} r^3\right) \qquad n = \frac{\text{\# Free Electrons}}{\text{Unit Volume}}$$

I am using MKS units, and I will use them throughout the course. If anyone wants to use CGS units, I might add, it is quite all right.

An electron that is moving out will have an average energy 1/2 KT. We set this potential times e = 1/2 KT.

$$\phi e = \frac{1}{2} KT = \frac{ne^2 r^2}{3\epsilon_0} \implies r^2 = \frac{3}{2} \frac{\epsilon_0 KT}{ne^2}$$

This is really quite straight-forward. I have just taken the potential and multiplied it by e which gives you the potential energy of an electron at a distance of r away from the sphere, and set the two energies equal. Then,

$$r = \left(\frac{3}{2} \frac{\epsilon_0 KT}{ne^2} \right)^{\frac{1}{2}}$$

This distance is the general size of the sphere in which you can have marked deviations from electrical neutrality.

There is a fundamental unit of length in plasmas called the Debye length. It is given by

$$\lambda_D = \left(\frac{\epsilon_0 KT}{ne^2} \right)^{\frac{1}{2}}$$

There is just the square root of three halves difference between the distance we calculated for a sphere and the Debye length. The difference between the two distances is due to geometrical factors; the Debye length is, if you work this out, instead of for a sphere, if you defined in terms of how far a particle can move away from a statistical fluctuation in a plane geometry.

The significance of the Debye length is that in a plasma over distances larger than the Debye length, you have electrical neutrality. The particles do not really have enough energy to be anything but electrically neutral inside of volume large compared to the Debye length. Also, if we put a wall, say a metal wall, next to a plasma where a charged probe, something like that...say a metal probe out here, then the influence of this will extend into the plasma about 1 Debye length again. The Debye length is not a magical, rigid number. It just gives one a feeling concerning a given plasma.

A plasma is made up of plus and minus charges. If you are concerned with volumes small compared to λ_D^3, the particles do not have the properties of a plasma; plasma concepts will not necessarily apply. For example, cosmic rays are not a plasma. Over the scale of the earth's magnetic field, they can be considered single particle trajectories.

Another property to consider is how fast the plasma can react to some sort of displacement; that is, if you perturbed the plasma, how fast would it react and come back to equilibrium? Say we have an infinite plasma, and we take a slab and, in this slab displace the plus charges one way and the minus charges the other. A charge separation results:

When you let go, the charges will bounce back. You will get a simple harmonic oscillator.

Another method of measuring a plasma's reaction times comes from the derivation of velocity of light. In deriving the velocity of light, you use the Maxwell equation:

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

There is a displacement current and a conduction current when a plasma is present.

If we have a sinusoidal electric field, then we can see that the displacement current (which has no inertia; that is, displacement current is a polarization current). If you examine it closely, it will go like this.....the phase of the displacement current will be such that it will lag the electric field (See Figure 1-1 PAGE 6). The conduction current I will be 180° out of phase with the displacement current.

For a sinusoidal oscillation, electrons are always 90° out of phase of the driving force; they lag behind, because of inertia, whereas the displacement current which is a polarization current, xxxxxx will lead the driving force by: 90°

Let's work this out:

$D = \epsilon_0 E \qquad E = E_0 \sin \omega t \qquad \frac{\partial D}{\partial t} = E_0 \epsilon_0 \omega \cos \omega t$

For the conduction current, i= nev We know the electron acceleration it is just e/m times the electric field, which is E_0 sin wt.

$$\frac{dv}{dt} = a = \frac{F}{m} = \frac{e E_0 \sin \omega t}{m}$$

Now, integrating this with respect to time, we find that the velocity is

$$v = -\frac{e}{m}\frac{E_0}{\omega}\cos \omega t \qquad \text{But } i = nev \text{ so,}$$

$$i = -\frac{ne^2}{m\omega} E_0 \cos \omega t$$

That is the conduction current in a plasma. As you might expect, since the conduction current is limited by inertia, as you go to very high frequencies, the conduction current approaches zero. For very high frequencies, electrons just can't move whereas displacement current increases with frequency.

Now we have the displacement current and conduction current, and we know that one is negative and the other is positive....as I said before, they are 180^0 out of phase. And this is something to ponder a little bit, because it is quite significant.

If we ask when will the two currents be equal and opposite; that is, if we ask when

$$i + \frac{\partial D}{\partial t} = 0$$

that will be the frequency for which there will be no electromagnetic propagation through the plasma. Some kind of current is needed to propagate electromagnetic waves. You need some place to store potential and kinetic energy. You either need a displacement current or a conduction current of some kind. We will come up to this again in hydromagnetic waves where we have transfers of energy from potential to kinetic energy.

Set these two currents equal and solve for w:

$$\epsilon_0 E_0 \omega \cos \omega t = \frac{ne^2}{m\omega} E_0 \cos \omega t \Rightarrow \omega_p = \left(\frac{ne^2}{m\epsilon_0}\right)^{1/2}$$

Wp is called the plasma frequency. This is a characteristic time. If you perturb a plasma with an electric field, say an electromagnetic wave, this is the characteristic time of reaction.

The name "plasma" was proposed by Langmuir and Tonks in a 1929 Physical Review paper. Plasma has the connotation coming earlier from medicine that plasmas were things that had to do with life, and they quivered. This medium quivers with the frequency Wp; hence, the name.

Obviously, a plasma isn't going to have this quivering property if the collision frequency of nucleons with themselves or a neutral gas is too high. If the collision frequency is large compared to Wp, then we don't have a plasma.

PROBLEMS

1. What is the Debye length in copper at room temperature? Can copper be considered a plasma?

2. What is the Debye length at 200 km altitude in the earth's ionosphere? (Use the tables given in the Satellite Environment Handbook. You will find that you have a choice of parameters, sunspot maximum, sunspot minimum, night and day. The parameters will vary. One thing you will get besides the Debye length at 200 km is practice in using this useful reference.)

3. What is the Debye length at 3 earth radii geocentric distance?
   (a) What is the Debye length for the Van Allen radiation?
   (b) And cosmic radiation?

(These problems are very simple. What we are trying to get at is when can one consider single particle trajectories and when cooperative plasma properties must be used.)

4. What is the relationship between the Debye length and the plasma frequency?

5. Derive the plasma frequency from the concept of a simple harmonic oscillator discussed earlier.

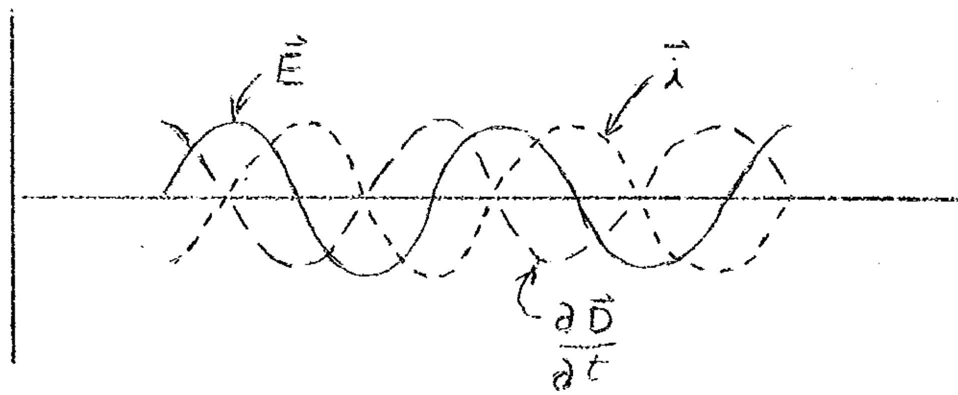


FIGURE I-1.

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SPACE SCIENCE 500

Lecture II

In a static uniform magnetic field,a charged particle is subject to the Lorentz force. In a magnetic field $\vec{B}$, a particle moving with the velocity $\vec{w}$ is subject to a force

$$\vec{F} = ze(\vec{w} \times \vec{B}).$$

($\vec{w}$ will be used for the particle velocity and $\vec{v}$ for the bulk motion of a plasma.) The force is always perpendicular to,or at right angles to the velocity vector, so this force can never change the energy of the particle.

Now balance the centrifugal force against the Lorentz force: Take a case where the motion is perpendicular to the magnetic field, then the centrifugal force is $\frac{mw^2}{a}$. The particle will move in a circle.

$$F = \frac{mw^2}{a} = ewB \Rightarrow a = \frac{mw}{eB}$$

a is the cyclotron radius. Sometimes it is called the radius of gyration which, I think, is an incorrect notation; radius of gyration should be reserved for pendulums and complex rotating bodies.

Furthermore, since the velocity of a particle is equal to the cyclotron radius times the angular frequency, we can solve for ω_c ;

$$w_\perp = a\,\omega_c \qquad \Rightarrow \qquad \omega_c = \frac{e}{m} B \, .$$

An easy number to remember, by the way, is that e/m for a proton in MKS units is 10^8. A unit you will be using a lot for magnetic fields is the gamma. One gamma is one millimicroweber per sq meter. Fields you will often be interested in are normally 10 gammas. This is roughly the field at about 10, or maybe 20 earth radii, or perhaps the interplanetary magnetic field.

Now, for a plasma let's define a stationary frame of reference as one in which the guiding center for a particle shows us motion relative to B. The center of the cyclotron motion is called the guiding center of the particle.

The stationary frame of reference this will define (which only works for a uniform magnetic field) is the one in which the guiding center is stationary. The stationary frame of reference is one in which there is no electric field perpendicular to the magnetic field, and, if there were one, the particle would move such that it would see no electric field.

Let's look at what happens to a particle. Say we have a magnetic field out of the plane of the page and a vertical electric field, and you have a particle there and you let go of it. It will start moving up along with the electric field, and then it will be turned by the magnetic field. The particle will have enough energy to come all the way back to the same potential from which it started. It will then stop and then go through the motion again and again.

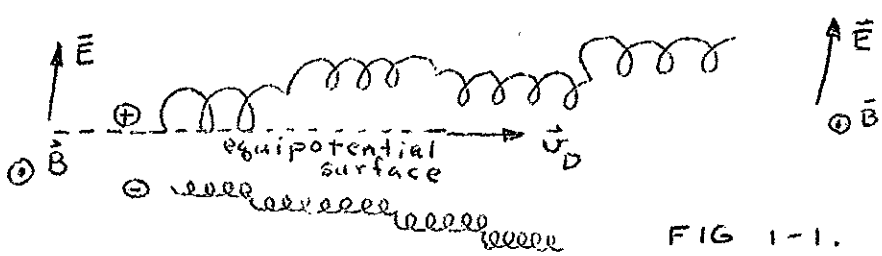

FIG 1-1.

The particle will move perpendicularly to both the electric and magnetic fields.

How fast will it move? One easy way of getting the answer is to say that the guiding center will not see any electric fields. You can usually transform into a frame of reference where you don't see the electric field. In cases applicable to the magnetosphere or interplanetary space, apparently you can always transform into a frame of reference where you don't see an electric field perpendicular to B. An electric field is produced by transforming the magnetic field.

$$\vec{E} = -\vec{v}_D \times \vec{B}$$

At the velocity $\vec{v}_D$ relative to the magnetic field, the electric field will counterbalance the applied electric field so that $\vec{E} + \vec{v} \times \vec{B} = 0$.

You can see immediately that if we apply an electric field perpendicular to a magnetic field, we are going to get a drift velocity which is

$$\vec{v}_D = \frac{\vec{E}}{|\vec{B}|} \times \vec{e}_0 \qquad \vec{e}_0 = \frac{\vec{B}}{|\vec{B}|} \; ; \quad \text{so} \quad |\vec{v}_D| = \frac{E}{B} .$$

$\vec{v}_D$ is the velocity of the guiding center; it is the average velocity for the particle. The motion shown in Figure 1-1 is a cycloid. The guiding center moving along like the axle of the wheel at a constant velocity and the rim of the wheel goes through these convolutions. You can get into the detailed particle motion -- it is more complicated, and you don't really gain much from it except in special cases where the guiding center approximation breaks down, but in most cases of interest, this is a good approximation.

The guiding center moves with the stationary frame; it defines the stationary frame in a uniform electric and magnetic field, and I stress that this is only true for uniform fields. Now this is for a positive particle. A negative particle will move in a motion mirroring that of a positive particle. It will move in the same direction with the same velocity independent of mass or sign of the particle -- it is just necessary to have some charge.

If we have a plasma and a magnetic field and apply perpendicular magnetic and electric fields, then the whole plasma will start moving as a body with the drift velocity E/B with just the proper velocity to transform away the applied electric field. If a plasma is moved through a static magnetic field, an electric field appears because the stationary frame of reference is one where the plasma is not moving. This electric field corresponds to

$$\vec{E} = -\vec{v} \times \vec{B} .$$

Now, a new subject -- conductivity in a plasma. We are touching all these things very briefly, but it will be enough so that when we talk about things like magnetic storms and solar winds, we can discuss what happens quantitatively. If you have a plasma, no magnetic field, and no collisions between particles -- and you apply an electric field, the particles will accelerate; they just keep accelerating and soon they are cosmic rays. This is a rather trivial result. Also, if you put on a magnetic field and apply an electric field parallel to B, the magnetic field has no effect. Since $\vec{F} = q\vec{W} \times \vec{B}$ and since $\vec{E}$ and $\vec{B}$ are parallel, the magnetic force is zero -- that, too, is not very interesting. A magnetic field is not significant if it is parallel to the electric field. The conductivity parallel to $\vec{B}$ is usually represented by σ_0 which is sometimes called the zero-field conductivity. The corresponding current is $\vec{j}_0 = \sigma_0 \vec{E}$. In the earth's magnetosphere, interplanetary, and galactic space there is always a magnetic field so any problem of interest will contain a magnetic field. Zero field conductivity for practical problems is the conductivity parallel to the magnetic field.

If we apply an electric field perpendicular to the magnetic field there is absolutely no current without collisions; the conductivity is zero.

The whole plasma, plus and minus charged particles, will drift together in the $\vec{E} \times \vec{B}$ direction with a velocity E/B. Now introduce a few collisions. The plus charges will tend to go along like that and every once in a while there will be a collision.

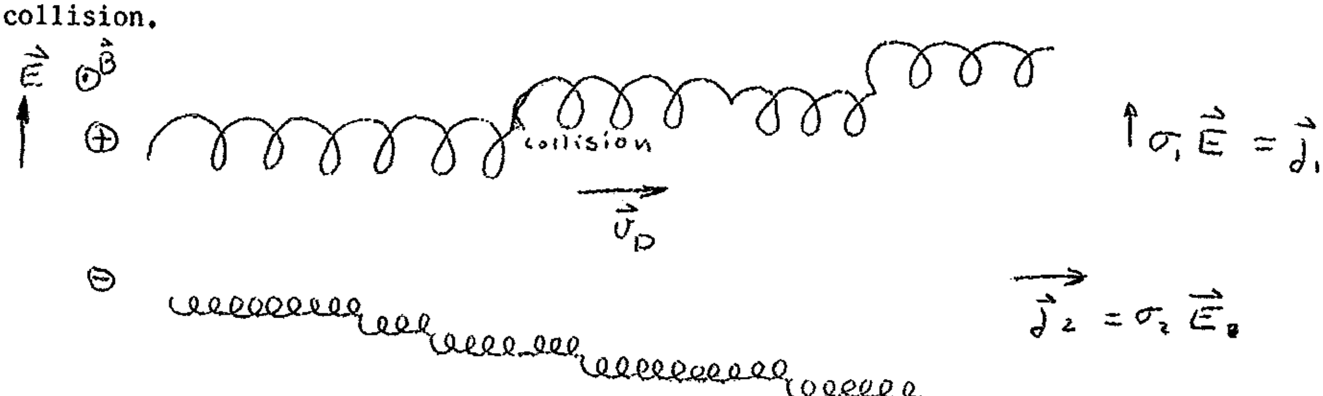

Figure 2-2

A current $\vec{j}_1$ results where $\vec{j}_1 = \sigma_1 \vec{E}$.

This is a current perpendicular to the magnetic field. This conductivity is obviously much smaller than σ_0 . σ_0 is usually enormous. There is one more conductivity, σ_2 , and we can see that as follows: (See Figure 2-2)

The plus charge has drifted to the right quite a bit more than the minus charge. So we also have a current $\vec{j}_2$ (see Figure 2-2), and that is called the Hall current. It comes about when you apply the electric field perpendicular to a magnetic field. The plus and minus charges try to drift as a body together; if there is something like a neutral gas present for collisions that inhibits one of the two components either plus or minus more than the other, $|\vec{j}_2| = \sigma_2 |\vec{E}|$ and is perpendicular to E and B. In the earth's ionosphere where there are big Hall currents, it turns out that the conductivity is negative--that is, the electrons do all the drifting and the protons are pretty well impeded by the neutral atmosphere. It is just the opposite of what is shown in Figure 2-2. The protons keep colliding and are rather well fixed. The electrons drift with the velocity that is some major fraction of the drift velocity; thus, the Hall conductivity is negative.

We have the three conductivities: σ_0, σ_1, σ_2 .

You can't have a σ_1 without having a σ_2 because you need collisions. If you want to see what they look like in general, look on pages 42 and 43 of the Satellite Environment Handbook for general formulas for σ_1 and σ_2 .

I should give you the names of these conductivities, too. σ_1 is called the reduced conductivity or the Peterson conductivity. This came about as people were calculating the current systems that should flow in the upper atmosphere and kept getting the wrong answer by factors like 10^6. Peterson pointed out that there was a magnetic field present, and this would affect the motion of the charged particle. They named the conductivity after him.

The Hall conductivity was named after Hall, the solid-state physicist who, many years ago, demonstrated this effect in a laboratory experiement.

The general expression for any kind of current that will flow in a plasma for any angle between the magnetic field and the electric field is given by

$$\vec{J} = \sigma_0 \vec{e}_0 [\vec{E} \cdot \vec{e}_0] + \sigma_1 [\vec{E} - \vec{e}_0 (\vec{E} \cdot \vec{e}_0)] + \sigma_2 [\vec{E} \times \vec{e}_0]$$

where $\vec{e}_0 = \dfrac{\vec{B}}{|\vec{B}|}$

Normally, any component of an electric field parallel to B is shorted out, but here you want to exercise some caution. As Dr. O'Brien has pointed out, the way it might be shorted out is to create an aurora or something very interesting. So, don't always assume that there are no electric fields parallel to B. No one has demonstrated that $\vec{E} \cdot \vec{B} \neq 0$, but no one has explained the aurora either.

The current for the case $\bar{E} \cdot \bar{B} = 0$ is

$$\vec{J} = \sigma_1 \vec{E} + \sigma_2 (\vec{E} \times \vec{e}_0) .$$

This is the form that you generally see in the literature. The assumption is usually made that there is no electric field parallel to B.

That takes care of conductivity.

Now I will tell you about power or rate of energy dissipation in plasma. Notice that if there is an electric field perpendicular to a magnetic field, a current will flow that is made up of $\vec{J}_1$ plus $\vec{J}_2$. Thus, in general $\bar{E}$ and $\vec{J}$ are not parallel.

Force times the velocity is power dissipation, i.e., the rate of change of energy since the force on a charged particle is in the E direction, $\vec{P} = \vec{E} \cdot \vec{J}$. There is no dissipation for the Hall current. Thus,

$$P = \sigma_1 E^2 = \vec{E} \cdot \vec{J} = (\vec{v} \cdot \vec{E})(ne) = \text{power dissipation/unit vol.}$$

$$\vec{F} \cdot \vec{v} = \text{power}$$

The electric field generally is hard to measure, but you can measure changes in magnetic field that can be related to currents. For example, you can fly a magnetometer through a current systems, and, from the measured changes in magnetic fields, determine the currents.

The total current is $$j^2 = E^2\sigma_1^2 + E^2\sigma_2^2$$

solving for E^2, $$E^2 = \frac{j^2}{\sigma_1^2 + \sigma_2^2}$$

We can thus derive the power dissipation in a form that is only a function of currents and conductivities.

$$P = \frac{\sigma_1 j_1^2}{\sigma_1^2 + \sigma_2^2} = \frac{1}{\sigma_3} j^2 \quad ; \quad \sigma_3 = \sigma_1 + \frac{\sigma_2^2}{\sigma_1}$$

σ_3 is called the Cowling conductivity.

If you look on pages 42 and 43 of the Satellite Environment Handbook, you will see that the conductivities are a function of the cyclotron frequencies for ions and electrons as designated by the sub-script e and i. There is another ω that is the applied frequency. Obviously, when you get to a very high frequency, if the collision time becomes very short compared to the applied frequency, the conductivities are going to be altered drastically because the particle would just vibrate back and forth many times between collisions. The collisions are obviously going to have a different effect (which is expressed in these formulas). I just want to point out that this discussion is for the case of uniform magnetic field, uniform electric field with no time variations or pressure gradients, etc.

HOMEWORK PROBLEMS

(1) In the formula for the drift velocity, what happens when

$E/B > c$? c = light velocity

(2) Draw the trajectories of electrons and protons in a plasma in thermal equilibrium showing the origin conductivities of sigma-1 and sigma-2, of the Hall current, and the Peterson current (individual trajectories showing the plus and minus charges).

(3) Take the region over the polar cap where the magnetic field is perpendicular to the surface of the earth and the ionosphere. For the polar regions B is perpendicular to the ionosphere. What is the angle between the electric field and the current? Let's say we just apply an electric field parallel to the earth, i.e., perpendicular to B. What is the angle between the electric field and the current that flows at 100 km and 200 km? Just look up conductivities in the Satellite Handbook and fill in the numbers.

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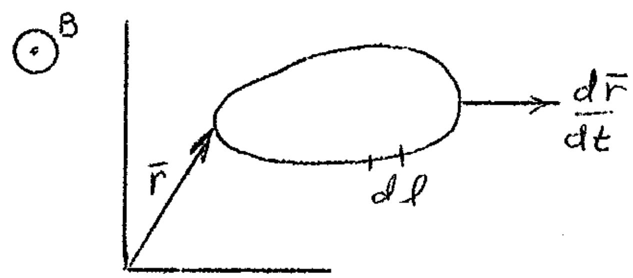
SPACE SCIENCE 500

Lecture III

We are going to talk about "frozen-in fields" - the terminology first suggested by Alfven. There is a very good discussion of this in Spitzer's book on page 40. Later on, for example, when we get to the Van Allen radiation, you will see that there are many conditions under which this concept doesn't work. It is good for uniform, slow-moving, and highly conducting plasma. The idea of a frozen-in flux is that if a highly conducting plasma is imbedded in a magnetic field, the plasma behaves as if it were glued to the magnetic field lines. If you try to change the magnetic field or move the plasma, the plasma will move in such a way that the field and the plasma stay together. It is as if you had superconducting loops. The flux through the superconducting loops will not change.

Let's take a two dimensional surface in the plane of the board, moving with a velocity $d\bar{r}/dt$ relative to a fixed coordinate system.



There is a magnetic field coming out of the board at some arbitrary angle that cuts through this surface. The flux through this surface is the surface integral of B, which we will say is the function of both r and time.

$$\phi = \int_{\text{Surface}} \bar{B}(\bar{r},t) \cdot d\bar{s}$$

B can change with the radial distance, and it can also change with time. ds is an element of surface area. If we integrate over the surface, we will get the total flux threading the surface.

Now we can write the rate of change of flux through this surface. There are two reasons it will change: (1) B is a function of r if the surface is moving and the field is not uniform in space. (2) Likewise, if B is a function of time, the flux through the surface would change even if the surface stood still. So, one can write two equations for the change of flux with time.

(1) $$\left(\frac{\partial \phi}{\partial t}\right)_1 = \frac{\partial}{\partial t}\int \bar{B}(\bar{r},t)\cdot d\bar{s}$$

(2) $$\left(\frac{\partial \phi}{\partial t}\right)_2 = \oint \bar{B}(\bar{r},t)\cdot\left(\frac{d\bar{r}}{dt}\times d\bar{l}\right)$$

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + (\vec{v}\cdot\nabla)\right)\int \vec{B}\cdot\vec{n}\,da$$

$$\frac{d\phi}{dt} = \frac{\partial}{\partial t}\left(\int \vec{B}\cdot d\vec{s}\right) + (\vec{v}\cdot\vec{\nabla})\int \vec{B}\cdot d\vec{s}$$

Here $d\bar{l}$ is a unit of the line element around the surface.

Now the total change in flux per unit time--the total derivative is

$$\frac{d\phi}{dt} = \frac{\partial}{\partial t}\int \bar{B}(\bar{r},t)\cdot d\bar{s} + \oint \bar{B}(\bar{r},t)\cdot\left(\frac{d\bar{r}}{dt}\times d\bar{l}\right)$$

We will use the approximation that we are always going to use in these problems, that E/B is less than the velocity of light.

We are going to assume an infinite, perfectly conducting plasma so that the plasma will always move such that

$$\bar{E} + \bar{v}\times\bar{B} = 0$$

Solving this for E and combining it with

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

we obtain

$$\frac{\partial\bar{B}}{\partial t} = \nabla\times\left(\bar{v}\times\bar{B}\right)$$

Substituting this into

$$\frac{d\phi}{dt} = \int_S \frac{d\bar{B}}{dt} \cdot d\bar{s} + \oint \bar{B} \cdot \frac{d\bar{r}}{dt} \times d\bar{l}$$

we obtain

$$\frac{d\phi}{dt} = \int_S (\nabla \times \bar{v} \times \bar{B}) \cdot d\bar{s} + \oint \bar{B} \cdot \frac{d\bar{r}}{dt} \times d\bar{l}$$

Interchanging the dot and cross products and applying Stokes' theorem to the secon d part of the right side of the equation we obtain

$$\frac{d\phi}{dt} = \int_S (\nabla \times \bar{v} \times \bar{B}) \cdot d\bar{s} + \oint_S (\nabla \times \frac{d\bar{r}}{dt} \times \bar{B}) \cdot d\bar{s}$$

$$\frac{d\phi}{dt} = 0 \quad \text{only if} \quad \frac{d\bar{r}}{dt} = \bar{v}$$

This means the rate of change of flux through this arbitrary surface will be zero if the surface moves with the plasma. Thus, if the flux doesn't change and we are moving with the plasma, then we can say that the flux lines move with the plasma. It is dangerous to say you have identified the flux line and these particular magnetic lines of flux move with the plasma. There are many cases where this is a handy concept, but there are also many cases where it doesn't work. It has its use, for instance, when you get an infinitely, highly conducting magnetic field stuck in a plasma. The real use of this concept is to think of highly conducting plasmas embedded in magnetic fields as being tied together.

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9/28/63

SPACE SCIENCE 500

Lecture IV

This morning we will discuss plasma pressure acting against a magnetic field. A magnetic field also can exert pressure which is

$$P = \frac{B^2}{2\mu_0}$$

This is either the magnetic field energy density or the magnetic pressure. Assume a unit area column of plasma moving with a velocity $\vec{v}$. If this column strikes a magnetic field, there is a momentum transferred every second, $P = (\rho \vec{v})\vec{v}$

That is, every second a column v cm long will strike the magnetic field and transfer a momentum ρv. If it strikes and doesn't bounce back, it just sticks. If the plasma bounces back, the pressure is $2\rho v^2$. If it scatters to the side, the pressure is increased by a factor of $\pi/2$, or something like that.

I am giving you the results first, and then I am going to demonstrate it. When the particle energy density or the plasma pressure exceeds the magnetic field energy density or pressure, the plasma will dominate and will push the magnetic field around -- the plasma will push the magnetic field aside, push it out of the way, carry it along -- anything the plasma wants to do, it will do and the magnetic field will just go along for the ride.

Now in the other case, when the magnetic energy is dominant, the magnetic field controls the plasma. For example, the Van Allen radiation trapped in the geomagnetic field, or mirror machines that contain a plasma. The magnetic field controls the plasma. This is a very powerful tool. All you have to do is to decide whether the particles or the fields are in control to get their energy densities. The one with the greatest energy density will dominate.

Let's take a very simple physical model to demonstrate the idea:

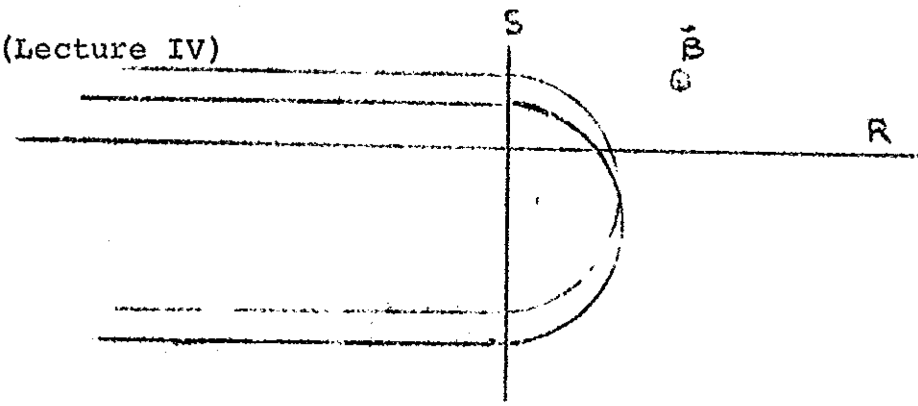


Fig. 4-1

A surface S; magnetic field $\vec{B}$ coming out of the paper; particles come thru S, turn around and go back out; and plus and minus charges of the same mass. As a first approximation, we will say the particles move along the arc of a circle with a cyclotron radius a.

Let us just pick another surface perpendicular to S: the surface R. Any positive particle that strikes within a distance 2a above R will pass thru R.

Now, what we have to do is relate the current $\vec{j}$ to the particles going thru S and R and then relate the particle flux to the magnetic field. The surface current density $\vec{j}$ is $\vec{j} = n_+ q_+ w_+ + n_- q_- w_-$

Any negative particle that strikes the surface S within a distance 2a below R must pass through R. Likewise, the positive particle that strikes within a distance of 2a above R must also pass thru R. If it is further away from R than 2a, it can't make it. So I know that the weaker $\vec{B}$ is, the bigger the current will be because the cyclotron radius a will be larger.

Right away you can kind of see how this is going to come out; there is an inverse relationship between this current and the magnetic field.

The surface current density is defined as the current flowing thru an area 1 meter out of the paper and whatever distance to the right of S is necessary to include the whole current. The surface current density is ne$\vec{w}$ times 2a. In other words, the flux of the particle striking the surface within a distance of 2a gives you the current.

Since $a = \frac{m\omega}{eB}$ and $J = ne\omega(2a)(1)$ = surface current density :

$$J_{surf} = \frac{2ne\omega m\omega}{eB} = \frac{2mn\omega^2}{B}$$

The current density is now in terms of the plasma energy density and the magnetic field. This current tends to cancel the magnetic field to the left of the current and reinforces the magnetic field on the other side. Now, let's ask the question "What current is required to just create a balance where it exactly cancels the field on the left and, conversely, doubles the field on the right?"

Since $\vec{\nabla} \times \vec{H} = \vec{J}$ take a line integral around J:

$$\oint \vec{B} \cdot \vec{dl} = 2B(L) = \mu_0 I$$

but $\frac{I}{L} = J_{surf} \implies J = \frac{2B}{\mu_0}$

Therefore, $J = \frac{2B}{\mu_0} = \frac{2nm\omega^2}{B} \implies$

$$\frac{B^2}{2\mu_0} = \frac{nm\omega^2}{2}$$

This is a condition for equilibrium where the plasma pressure is just barely balanced against the magnetic field. When the magnetic field has the greater energy density, it will push the plasma around; if the plasma has the greater energy density, it will push the field around. This is the fundamental idea that will come up again and again. This is the mechanism.

Now, we want to talk about magnetic diffusion thru a plasma, and I am afraid we won't get all the way thru today.

If we start with Maxwell's equation

$$\bar{\nabla} \times \bar{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t}$$

and take the curl

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{B}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{B}) - \nabla^2 \bar{B}$$

$$\nabla^2 B = -\mu_0 \vec{\nabla} \times \vec{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \bar{\nabla} \times \bar{E}$$

?

In order to solve this equation we have to know the functional relationship between $\vec{E}$ and $\vec{j}$. As we showed earlier in the discussions of conductivity, $\vec{E}$ and $\vec{j}$ are not necessarily parallel. The angle between $\vec{E}$ and $\vec{j}$ depends on $\sigma_\parallel$ and $\sigma_\perp$. This makes the problem very messy. The problem would be greatly simplified if $\vec{E}$ and $\vec{j}$ were parallel, i.e., $\vec{j} = \sigma \vec{E}$
So, first, let's get a relationship between $\vec{E}$ and $\vec{j}$, and then come back to this equation and put in whatever we come up with.

Let's take another model: We have a uniform magnetic field and in it we will put in a bubble of plasma such that the nkT, the energy density of the plasma, exceeds the magnetic field density initially. The field lines will be bowed out as shown in Fig. 4-2.

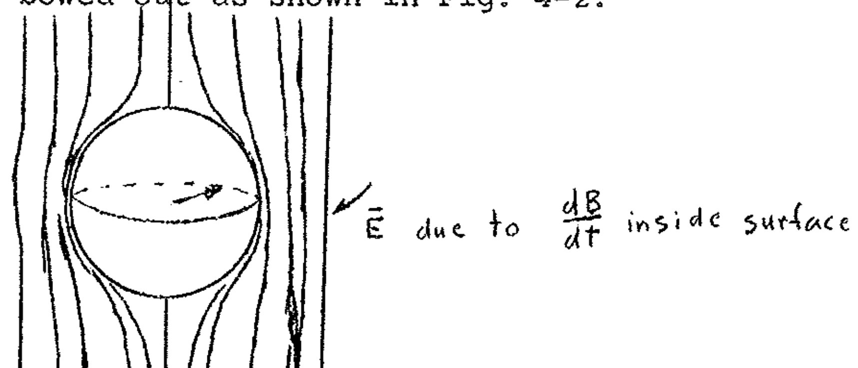


Fig. 4-2
Spherical Bubble

What I have here is a bubble of plasma, shown by the dotted line, and a uniform magnetic field that has been pushed aside by this plasma bubble. A surface S is drawn thru the bubble and is perpendicular to the magnetic field. After we put the bubble in and let go, the magnetic field lines start to move in toward a configuration of minimum energy--which is one in which the magnetic field lines are all parallel. An electric field $\vec{E}$ is around the bubble because there is a $d\vec{B}/dt$ as the field lines move into S. Call this $\vec{E}$.

Fig. 4-3 shows S redrawn so that S is in the plane of the paper, i.e., the whole configuration is turned 90°, and $\vec{B}$ is out of the paper.

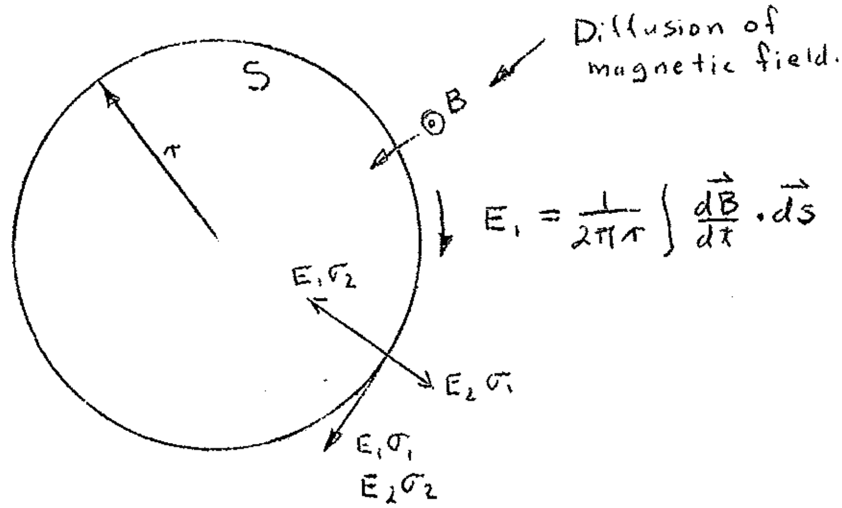


Fig. 4-3

$\text{curl } E = -\frac{\partial B}{\partial t}$

$\int E \cdot dl = -\int \frac{\partial B}{\partial t} \cdot ds$

$E \cdot 2\pi r = -\int \frac{\partial B}{\partial t} \cdot ds$

If we say this radius is $\vec{r}$ then

$$\vec{E}_1 = \frac{1}{2\pi r}\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

E_1 produces a current, $j_1 = \sigma_1 E_1$ and a Hall current directed towards the center, $j_2 = \sigma_2 E_1$. But we are going to get this Hall current all the way around. In other words, we have a complete radial flow, either in or out (depending on whether σ_2 is plus or minus), and we have a divergence of charge in the middle. This can't go on. A polarization field E_2 will be built up so that

$$\sigma_1 E_2 = \sigma_2 E_1$$

There is no net current flowing toward or away from the center in a steady state. We can solve for E_2: $E_2 = \frac{\sigma_2}{\sigma_1} E_1$

E_2 also causes a Hall current which is $\sigma_2 E_2$.

All the σ_2 are the same. They just represent the conductivity in a direction perpendicular to the electric and magnetic field.

We are about to come up with something exciting. Summing the two currents, $j_{total} = \sigma_2 E_2 + \sigma_1 E_1 = (\sigma_1 + \frac{\sigma_2^2}{\sigma_1}) E_1 = \sigma_3 E_1$

$$\vec{j}_{total} = \sigma_3 \vec{E}$$

Not only is $\vec{j}_{total}$ related to a conductivity that we are already familiar with, but $\vec{j}$ and $\vec{E}$ are parallel. It is just what we wanted. We probably couldn't have solved the problem otherwise. I am not sure we can now, but at least there is a chance.

HG

David Cummings

9/30/63

SPACE SCIENCE 500

Lecture V

We have shown that the electric field and the current are parallel even though there is a finite σ_1 and σ_2, a rather amazing result. The equation we wanted to solve was

$$\nabla \times \bar{B} = \mu_0 \bar{j} + \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}$$

If we take the curl of both sides and substitute $\bar{j} = \sigma_3 \bar{E}$,

(1) $$\nabla \times (\nabla \times \bar{B}) = \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = \mu_0 \sigma_3 (\nabla \times \bar{E}) + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \times \bar{E})$$

The term $\nabla(\nabla \cdot \bar{B}) = 0$. Substituting $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ into equation (1) and multiplying by minus one we obtain

(2) $$\nabla^2 \bar{B} = \mu_0 \sigma_3 \frac{\partial \bar{B}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2}$$

Once we have shown that the current and the magnetic fields were parallel (just the standard Ohm's law type conductivity), a wave equation and a diffusion equation result. The diffusion term is $\mu_0 \sigma_3 \frac{\partial \bar{B}}{\partial t}$ and the wave term is $\frac{1}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2}$. The question arises as to which of these terms dominate in conditions of geophysical significance.

What we did here was pretty much the standard derivation for the velocity of light. If the conductivity is zero, we get just the wave equation for the velocity of light in a vacuum.

If we take a time-periodic case where we assume that

$$\bar{B} = \bar{B}_0 e^{i\omega t}$$

and substitute this equation for B into equation (2), we find that the diffusion term is important when $\frac{\omega \epsilon_0}{\sigma} << 1$, where ω is the wave frequency.

Let's take the case where the conductivity is sufficiently high that $\frac{\omega \epsilon_0}{\sigma} << 1$, and substitute $\bar{B} = \bar{B}_0 e^{i\omega t}$ into equation (2), so that

$$\nabla^2 \bar{B} = \mu_0 \sigma_3 i \omega \bar{B}_0$$

The solution to this equation is

$$B = B_0 e^{-\sqrt{\mu_0 \sigma i \omega}\, r}$$

where r is just the radial distance into the conductor.

stet... What we have derived is the ~~radio~~ skin depth. This is what the skin depth means; in the time of one cycle, or some fraction of a cycle, given a plasma with some conductivity, the magnetic field will diffuse into it one skin depth. The formula for skin depth is ordinarily given as

$$\delta = \frac{1}{\sqrt{\frac{\omega}{2}\mu_0 \sigma}}$$

We are now going to talk about hydromagnetic waves considering an all prevailing plasma in a magnetic field (I will use the concept of frozen-in flux that we talked about earlier where the plasma is tied to the field lines). I take a field line, draw it up, and let go, then the analogy is that of the guitar string. The magnetic field tends toward a condition of minimum energy which is when the field lines are straight and uniform. Faraday used to think of magnetic field lines as stretched rubber bands, and that is a very useful concept. Thus, the plasma loaded magnetic field will vibrate like a stretched string; waves can travel along the magnetic field lines.

Another way to generate a wave is to push the field as shown in Fig. 5-1 and generate essentially a sound wave.

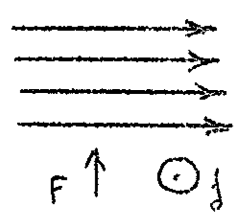


Fig. 5-1

If we use the condition where $\frac{B^2}{2\mu_0}$ is larger than nKT, that is, the particle pressure is negligible and the magnetic pressure determines the restoring force. The result is a hydromagnetic wave.

The transverse wave propagating down the field line is the one that Alfven originally discovered in his first publication. This is called V_A for the Alfven velocity. The other mode, also shown above, propagates with the velocity if $B^2/2\mu_0 >> nkT$. Now they are all referred to as hydromagnetic or Alfven waves.

Alfven's contribution is really amazing because, if you look at what we just did, we first derived the propagation of waves in a conductor and found they would penetrate only about one skin depth. If the conductivity goes to infinity, there is no penetration at all; that is, it might be said that there can be no electromagnetic wave propagation in a perfectly conducting fluid. Yet, that is not true. Once you see it, it is rather obvious and simple that you can propagate, under certain conditions, electromagnetic waves in perfect conductors -- in fact, there are several ways, and this is one of them. But it required special conditions, and Alfven was able to find these conditions and demonstrate them. It is one of the big break-throughs in astrophysics and space science in general.

Let's do a derivation just to show that these are electromagnetic waves that have many properties of an ordinary electromagnetic wave but that can propagate through a medium of infinite conductivity. Let's start out with a uniform magnetic field and embedded in it is an infinitely conducting plasma of density ρ

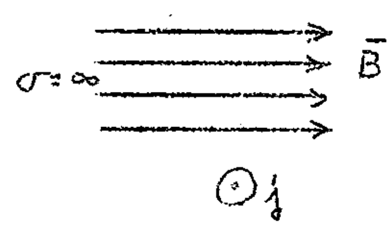


Fig. 5-2

If a current j is run perpendicularly to B as shown, then we know that there will be a force of (3) $\bar{F} = \bar{j} \times \bar{B} = ma = \rho \frac{d\bar{v}}{dt}$

This is one of the auxiliary equations that defines the properties of the medium.

Another auxiliary equation is $\bar{E} + \bar{v} \times \bar{B} = 0$. In other words, if an electric field is applied, the plasma will move in a direction $\bar{V} \times \bar{B}$. The motion of the guiding center is such as to cancel the applied electric field.

One other condition is that, since it is an infinitely conducting plasma, $\bar{E} \cdot \bar{B} = 0$, so we only need to worry about components of $\bar{E}$ perpendicular to $\bar{B}$. Taking the time derivative of $\bar{E}$, we get

(4) $$\frac{\partial \bar{E}}{\partial t} = -\bar{v} \times \frac{\partial \bar{B}}{\partial t} - \frac{\partial \bar{v}}{\partial t} \times \bar{B}$$

One condition I was mentioning was that the background field is very strong. We are only going to perform small perturbations on a very strong, uniform field. The velocity, as the field lines move back and forth, will go from zero to some small finite velocity and back to zero. Where the magnetic field is strong, there will be a big background field and small changes from it. Now, the $\partial \bar{v}/\partial t$ and the $\partial \bar{B}/\partial t$ terms are about the same order of magnitude. But $\bar{B}$ is very large because of the strong background field and $\bar{v}$ is usually close to zero. We have no initial velocity, just small perturbation from an initial state. The term $\bar{v} \times \frac{\partial \bar{B}}{\partial t}$ is negligible compared to $\frac{\partial v}{\partial t} \times \bar{B}$ because there is a strong background field, $\bar{B}$, and there is no steady velocity.

Now we can take $\frac{\partial \bar{v}}{\partial t} = \frac{1}{\rho}(\bar{j} \times \bar{B})$ from equation 3 and put it in equation 4.

(5) $$\frac{\partial \bar{E}}{\partial t} = -\frac{\partial \bar{v}}{\partial t} \times \bar{B} = -\frac{1}{\rho}(\bar{j} \times \bar{B}) \times \bar{B}$$

If you work out the various vectors on how things go, you can find out that j is parallel to $+\frac{\partial \bar{E}}{\partial t}$ rather than at some angle to the electric field. Also, $\bar{j}$ must be perpendicular to B. Therefore, we can drop the vector notation and can write

(6) $$\frac{\partial E}{\partial t} = \frac{1}{\rho} j B^2$$ when j IS PARALLEL TO $+\frac{\partial E}{\partial t}$ AND PERPENDICULAR TO $\bar{B}$.

In this case, you want to solve for the current $\bar{j}$ which is

(7)
$$j = \frac{\rho}{B^2}\frac{\partial E}{\partial t}$$

Now we can put the vector notation back in because we have already found that $\bar{j}$ and $\frac{\partial E}{\partial t}$ are parallel.

Now we are in a position to proceed with the derivation of the velocity of electromagnetic waves in a plasma. Starting with Maxwell's equation

(8)
$$\nabla \times \bar{H} = \frac{1}{\mu_0}(\nabla \times \bar{B}) = \bar{j} + \frac{\partial \bar{D}}{\partial t}$$

Rewriting Maxwell's equation in terms of $\bar{B}$ and $\bar{E}$ and substituting equation 7 for the current $\bar{j}$, we find

(9)
$$\nabla \times \bar{B} = \frac{\mu_0 \rho}{B^2}\frac{\partial \bar{E}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

Now another amazing thing has happened -- we have gotten the conduction current in phase with the displacement current.

This is truly remarkable in that something we did to the plasma brought the two currents into phase. We just went through deriving the conditions for an electromagnetic wave in a conductor where

(10)
$$\bar{j} = \sigma \bar{E}$$

and the conduction current was 90° out of phase with the displacement current. There is all the difference in the world, well, there is 90° difference between those two. Equation 10 leads to a diffusion term rather than contribute to the wave propagation.

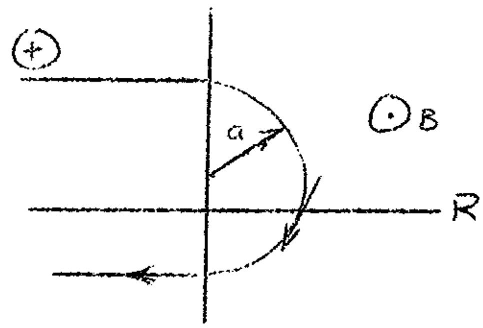
If we take the curl of both sides of equation 9 and multiply by minus one, we find that

$$\nabla^2 B = \frac{\mu_0 \rho}{B^2}\left(\frac{\partial^2 B}{\partial t^2}\right) + \frac{1}{c^2}\left(\frac{\partial^2 B}{\partial t^2}\right)$$

Both of these terms are wave equation terms. We get a wave velocity which will be called the hydromagnetic wave velocity which is

$$V_{hm} = \frac{1}{\left[\frac{\mu_0 \rho}{B^2} + \frac{1}{c^2}\right]^{\frac{1}{2}}}$$

HOMEWORK

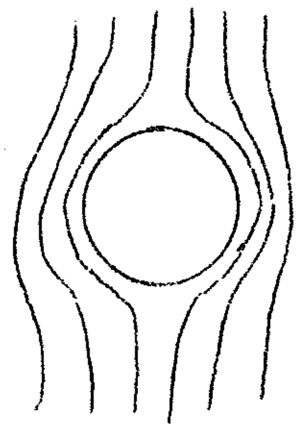


Problem 1:

Take into consideration the fact that the particles crossing the R plane do not cross perpendicular to it. How does this effect the current?

Problem 2:

If we have a sphere of plasma in a magnetic field and the sphere is enlarging continuously with time



a. Does a current flow?
b. Is $E \cdot j = 0$
c. If it is not zero, where does the energy go?

Problem 3:

From the data in the Satellite Environment Handbook determine whether the diffusion equation or the wave equation will dominate at 100 km altitude and at 300 km.

Problem 4:

In the derivation of hydromagnetic waves, what did we assume or do to the plasma to bring the current and $\frac{d\bar{D}}{dt}$ into phase. That is, why did a wave equation result rather than a diffusion equation?

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10/2/63

SPACE SCIENCE 500

Lecture VI

Last time we obtained for the velocity of an electromagnetic wave in a plasma,

$$V = \frac{1}{\left[\frac{\mu_0 \rho}{B^2} + \frac{1}{c^2}\right]^{1/2}}$$

Now, if we have a sufficiently high density and a magnetic field present (a large magnetic field) so that $\mu_0 \rho / B^2$ is much greater than c^2, then we can neglect $1/c^2$ and write the hydromagnetic wave velocity

$$V_{hm} = \frac{B}{\sqrt{\mu_0 \rho}}$$

Both a magnetic field and a plasma must be present. The form of the equation turns out to be rather fortunate for most applications in space science. $\overline{B}$, the magnetic field such as the magnetic field of the earth, is usually known in most places to . a factor of 2 or so. The density you know perhaps to a factor of 10, but, since it is the square root of the density that matters for V_{hm}, that means there is only a corresponding error of 3 in the velocity. The thing we don't know very accurately is only to the ½ power, and the thing we know more accurately is the dominant term. Generally, you can define the hydromagnetic wave velocity within a factor of 10 or a 100 in most regions of space.

In the derivation, there was no stipulation made as to the direction of propagation relative to the magnetic vector. This derivation applies for electromagnetic propagation or hydromagnetic wave propagation in any direction relative to $\overline{B}$.

One condition for propagation of a hydromagnetic wave is that there must be a magnetic field. If $\overline{B}$ goes to zero then, from

$$\rho \frac{\partial \overline{E}}{\partial t} = (\overline{j} \times \overline{B}) \times \overline{B}$$

we note that $\rho \frac{\partial \overline{E}}{\partial t}$ is zero. We know that ρ is not zero, therefore $\partial \overline{E}/\partial t$ is zero. The medium just doesn't move. The whole derivation no longer applies.

Another condition is high conductivity. We have already seen that there is magnetic diffusion in a plasma, and if we have a poor conductor, the plasma won't be frozen to the field and you won't really get the hydromagnetic waves.

Another condition that we didn't talk about at all is that the wave frequency ω has to be less than ω_c. That is, the applied frequency must be less than the cyclotron frequency. The reason for this is that we have a particle going around the magnetic field lines, and if we move the magnetic field appreciably in a time less than the cyclotron period, the ion will not be able to keep up with the motion of the magnetic field. That is, if you send optical radiation (light) through a plasma, the ions are too heavy to move, they are too slow to follow these fast perturbations.

One condition in which frozen-in flux doesn't apply is for very rapid changes.

The last condition (for this particular derivation) is that the particle pressure be negligible compared to the magnetic field pressure. The forces that we considered are all magnetic -- we don't have any pressure gradient forces which should have been included if we are going to include particle pressure. If we were going to include particle pressure in the simplified derivation that I gave you, we would have to include pressure gradient forces in the initial equations.

If nkT is less than $B^2/2\mu_0$, then a hydromagnetic wave will propagate in the $\bar{B}$ direction only if it is a sheer wave.

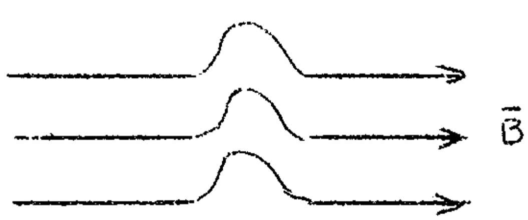


Fig. 6-1

SHEER WAVE

I am going to go through one more derivation for hydromagnetic waves that is quite independent of the previous one and again different from what you will find in Spitzer. It will give you more physical insight into what the hydromagnetic wave is. The first derivation showed that it was a form of electromagnetic propagation, a very special form of electromagnetic propagation where an electromagnetic wave could propagate through a perfect conductor.

Take a uniform magnetic field and compress it (as shown in Fig. 6-2) with a superconducting piston.

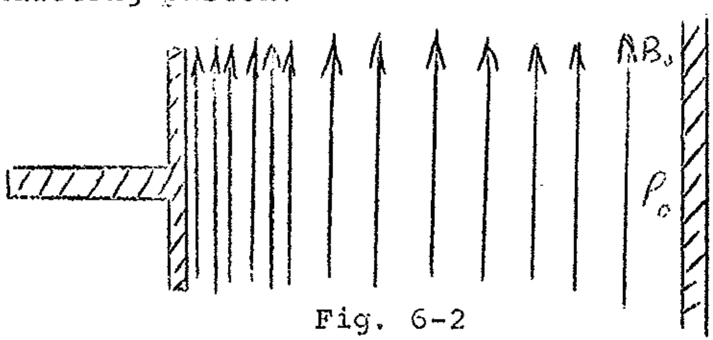


Fig. 6-2

This compression will propagate through the plasma. The magnetic field will compress locally first and this compression will propagate into the plasma much like a sound wave. As a very general proposition for the propagation of longitudinal waves like this, you can write the velocity of propagation as

$$V = \left(\frac{dP}{d\rho}\right)^{1/2}$$

If you have a medium with a compressibility and compress it suddenly, this phenomenon (the compression or the signal) will propagate through with this velocity.

If we take the condition that

$$\frac{B^2}{2\mu_0} >> nKT$$

then the pressure is just a magnetic pressure, $B^2/2\mu_0$. That is, all the restoring forces of the medium are due to magnetic pressure. The mass of this system is the mass of the particles. If we let the mass density go to zero, by the way, there is still an energy mass left (relativistic). You have a certain energy density, and since $E = mc^2$ and you can assign an m by equating the energy in the magnetic field to a mass, velocity of light can be derived.

The mass density is the ρ of the plasma. If we have an infinite conductivity, the plasma is frozen into the magnetic field so that B/ρ is equal to a constant. $\frac{B}{\rho}$ = constant

When we squeeze the field, the space between the field lines decreases, and the field strength increases proportionately. The space in between ions decreases, and the ion density increases proportionately. But notice something interesting here -- you don't get any change in the particle spacing in the direction of $\overline{B}$ (See Fig. 6-2). There are really only 2 degrees of freedom shown in this model. There is no coupling between the degree of freedom in the direction of $\overline{B}$, and the 2 degrees of freedom in the plane perpendicular to $\overline{B}$. The magnetic field pressure is hydrostatic. If you compress the magnetic field, the field lines will arrange themselves so the spacing between field lines in the plane perpendicular to the compression is equal. Since the plasma is frozen to the fluid, then B/ρ is a constant.

The velocity of the disturbance is not the velocity of the piston. They are not necessarily related -- the amplitude of the disturbance is related to the velocity of the piston, but not the velocity of the disturbance.

This field is B_o before it is perturbed and the density is ρ_o, and since the ratio is a constant, we can write $B = \frac{B_o}{\rho_o}\rho$. Therefore, the magnetic pressure $\frac{B^2}{2\mu_o}$ is $P = \frac{B_o^2\rho^2}{2\mu_o\rho_o^2}$

Now all I have to do is take the partial of P with respect to ρ to get the velocity:

$$V^2 = \frac{\partial P}{\partial \rho} = \frac{B_o^2\rho}{\mu_o\rho_o^2}$$

Since the perturbation is small, you can write that $\rho = \rho_0 + \Delta\rho \cong \rho_0$ If $\Delta\rho << \rho$

$$V^2 = \frac{B^2}{\mu_0 \rho}$$

Now you should have a very good feeling for this as a form of sound wave. And we can even do better than that.

If we wanted to look at this as a modified sound wave, you can find that the velocity of a sound wave is (1) $V = \left(\gamma \frac{P}{\rho}\right)^{1/2}$ where gammais the ratio of specific heats. $\gamma = C_p/C_v$.

Before Laplace, people used to use $\left(\frac{P}{\rho}\right)^{1/2}$ which is Newton's result, for the sound velocity and then wondered why the theory and the experiment didn't match. The 15% or so error they made was due to the fact that when a gas is compressed, it heats up, the rms molecular velocity goes up, and the signals propagated faster than expected.

We know γ here, because I mentioned earlier that there was one degree of freedom that in this model doesn't interact with the other two. There are really two degrees of freedom that we are concerned with -- the third degree of freedom is of no consequence in this problem. The energy and the velocity, and the density in this direction could be anything and it wouldn't affect the problem, and it won't change when you compress the gas.

Gamma is given by a general thermodynamic argument -- it is the number of degrees of freedom plus two divided by the number of degrees of freedom. $\gamma = \frac{f+2}{f}$

Here we have said that we are only concerned with 2 degrees of freedom, so 2 + 2 divided by 2 is 2 -- and gamma is equal to 2 instead of the usual 5/3rds. I might mention that I don't think gamma is equal to 2 in real cases that you run into in space science. The model shown in Fig.6-2 is very ideal, ideal where there is no coupling between modes, there is no turbulence, no radiation, and no instability. I don't think that ever happens. In real cases, I think you ought to use gamma of the order of 5/3 (although I can't prove it).

Let's go and put this gamma into equation 1.

$$V = \left(2 \times \frac{P}{\rho}\right)^{1/2} = \left(2 \times \frac{B^2}{2\mu_0 \rho}\right)^{\frac{1}{2}} = \frac{B}{\sqrt{\mu_0 \rho}}$$

Up to now we have assumed that the plasma pressure was insignificant, i.e., we used only the plasma mass-loading and the magnetic pressure. Now let's take the case where we consider the plasma pressure is comparable to the magnetic pressure and go through this again. The total pressure is

$$P = B^2/2\mu_0 + nKT$$

or, since $n = \frac{\rho}{m}$, $P = \frac{B^2}{2\mu_0} + \frac{\rho KT}{m}$

The wave velocity, as before, is $V = \left(\frac{dP}{d\rho}\right)^{1/2}$. It is well to note that ρ and T are related in an adiabatic compression by the relationship $T\rho^{(1-\gamma)/\gamma}$ = const. Solving for $\left(\frac{dP}{d\rho}\right)^{1/2}$ we obtain

$$\left(\frac{dP}{d\rho}\right)^{1/2} = V = \sqrt{\left(\frac{B_0}{\sqrt{\mu_0 \rho}}\right)^2 + \left(\sqrt{\frac{\gamma KT}{m}}\right)^2}$$

If the plasma energy density is greater than the magnetic field energy density, V will be equal to the ordinary sound propagation velocity and if the magnetic field dominates, we get the hydromagnetic wave velocity. When the energy densities are nearly equal, the resulting wave generally goes by the name of magneto-accoustic, which is an obvious terminology.

There are several other hydromagnetic wave names with which you should be familiar. As we have shown the Alfven wave (the transverse or sheer wave that propagates parallel to B) and the compressional wave (the longitudinal or sometimes modified sound wave that propagates perpendicular to B), both have the same velocity, $V_{hm} = \frac{B}{\sqrt{\mu_0 \rho}}$

if $\frac{B^2}{2\mu_0} \gg nKT$

This wave velocity is sometimes called the "fast wave"; note it is isotropic. There is another wave that is called the "slow wave." Its velocity is $V_s = V_{hm} \cos\theta$ where θ is the angle between the wave crest normal and $\bar{B}$. Actually, the slow wave is really a pure Alfven wave generated with oblique wave crests as shown in Fig. 6-2. The energy

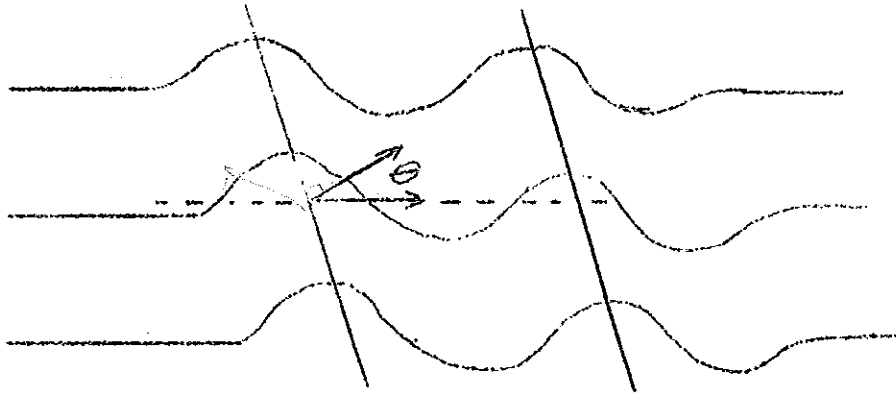


Fig. 6-2

of the wave (the Poynting's vector) is actually propagating in the direction of B, but the obliquely displaced wave crests give the appearance of a wave propagating at an angle θ to the $\overline{B}$ field. Thus, a polar diagram for the velocity of the slow and fast wave may be drawn as in Fig. 6-3

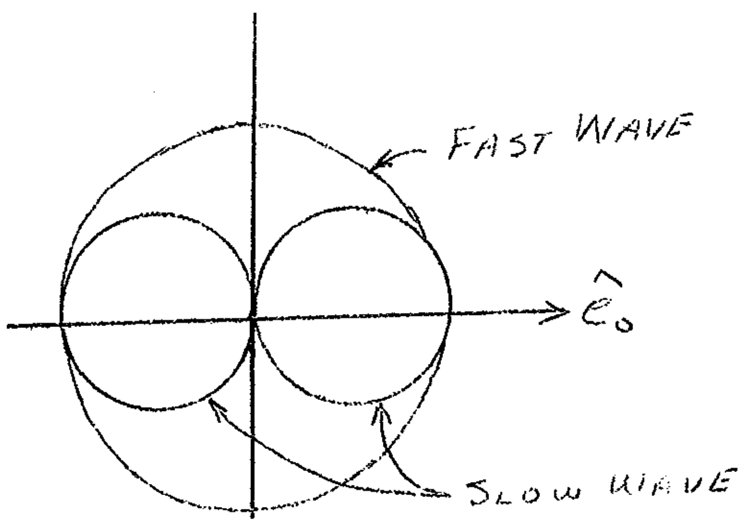


Fig. 6-3

Now we will discuss magnetic diffusion times for a couple of particular cases. The last time we talked about magnetic diffusion, we obtained

$$\nabla^2 B = \mu_0 \sigma_3 \frac{\partial \overline{B}}{\partial t}$$

The time dependent solution of this equation depends upon the initial condition; there isn't any single answer for this differential equation. But, if you specify the initial conditions, you can always give the time dependent solution to describe how an initial B configuration through space will decay away.

Let's take two simple cases and show how this might work. We will get a couple of answers that will give a feeling for how magnetic fields decay over a given scale length. For an initial condition, let's assume that we have

$$B_i = B_0 e^{-r/\ell}$$

ℓ is the distance over which B drops to 1/e of its maximum value. We set up this field configuration and/then turn off whatever the forces are that maintain this distribution. $\vec{B}$ decays away as the energy of $B^2/2\mu_0$ is dissipated in the form of $\bar{E}.\bar{j}$.

This equation is satisfied by $\bar{B} = \bar{B}_i e^{-\frac{t}{\mu_0 \sigma_3 \ell^2}}$?

The characteristic time for decay is

$$\tau_1 = \mu_0 \sigma_3 \ell^2$$

This is a very interesting result because, for one thing, it shows that we do have diffusion because τ is proportional to ℓ^2 -- diffusion is like a random walk.

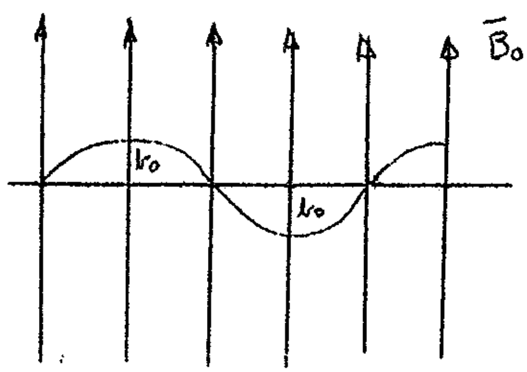
David Cummings

10/1/63

SPACE SCIENCE 500

Lecture VII

Assume a uniform magnetic field modulated with a small wave vector $\vec{b}_o$ sinusoidally so that a hydromagnetic wave propagates.



$B = B_o + b_o \sin wt$

Fig. 7-1

From the Maxwell's equation, $\bar{\nabla} \times \bar{B} = \mu_o \vec{j}$ (neglecting the displacement current now; we are in the hydromagnetic wave mode completely) and by taking a line integral around the surface, a, b, c, d,

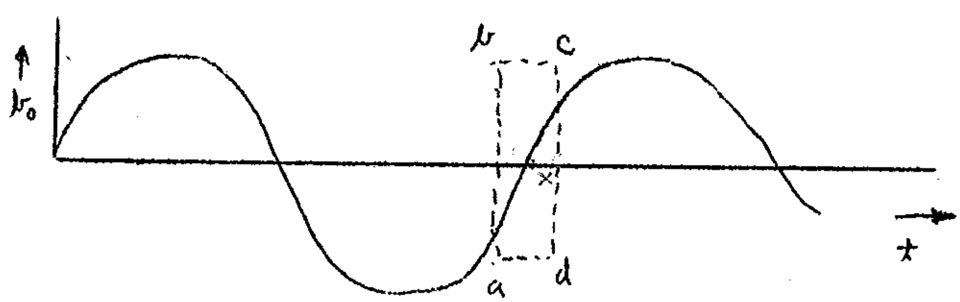


Fig. 7-2

it can be shown that the maximum current for a sinusoidal hydromagnetic wave is

$$j_o = \frac{2\pi b_o}{\mu_o \lambda}.$$

current density

Now we will study hydromagnetic wave attenuation. Assume medium that is slightly resistive, that is, the conductivity is not infinite, and let's see how far the waves can propagate before resistance, operating on the current, makes the waves fade away.

$$P = \frac{j_{rms}^2}{\sigma_3} = \frac{j_0^2}{2\sigma_3}$$

Substituting $j_0 = \frac{2\pi b_0}{\mu_0 \lambda}$ into $P = \frac{j_0^2}{2\sigma_3}$ we obtain

$$P = \frac{2\pi^2 b_0^2}{\sigma_3 \mu_0^2 \lambda^2}$$ (power dissipation per unit volume).

The energy stored per unit volume in a hydromagnetic wave is

$$W = \frac{b_0^2}{2\mu_0}$$

To get the time for a wave to decay to 1/e of its initial amplitude, one simply divides energy in the wave by the rate of power dissipation.

$$\tau = \frac{W}{P} = \frac{b_0^2}{2\mu_0} \cdot \frac{\sigma_3 \mu_0^2 \lambda^2}{2\pi^2 b_0^2} = \frac{\mu_0 \sigma_3 \lambda^2}{4\pi^2}$$

In Lecture VI, we showed that the time for a magnetic field to diffuse a distance ℓ through a plasma was given by $\mu_0 \sigma_3 \ell^2$. If we let $\ell = \frac{\lambda}{2\pi}$ we may think of attenuation of hydromagnetic waves as wave troughs and crests diffusing into each other and annihilating each other.

<u>RADIATION PRESSURE</u> - Assume a background field $\bar{B}_o$ and modulate it with a little field, $\bar{b}_o$, which is a sine wave so that $\bar{B} = \bar{B} + \vec{b}_o \sin \omega t$. The radiation pressure is the energy densi-

due to b_o; energy density and radiation pressure are the same things--a force per unit area is the same as energy per unit volume.

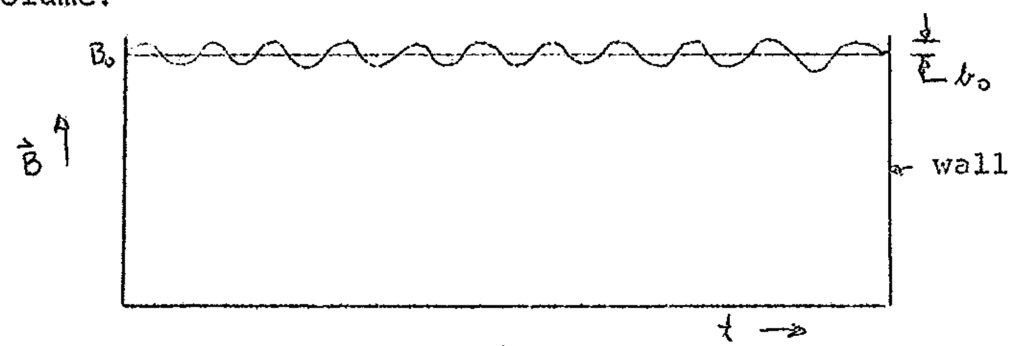


Fig. 7-3

$\bar{B} = \vec{B}_o + \vec{b}_o \sin wt$; $b_o = \sqrt{2} b_{rms}$

$$\frac{B^2}{2\mu_o} = \frac{B_o^2}{2\mu_o} + \frac{B_o b_o}{\mu_o} \sin wt + \frac{b_o^2}{2\mu_o} \sin^2 wt$$

Now we have to average this over one complete cycle to find out what the average energy density is, i.e., the average pressure against the wall (see Fig. 7-3). The average isn't zero: the average of sine squared over one cycle is ½. You know this because sine squared plus cosine squared is equal to one. The average of each of these must be ½ because ½ + ½ = 1.

$$\left[\frac{B^2}{2\mu_o}\right] \text{average} = \frac{B_o^2}{2\mu_o} + \frac{b^2_{rms}}{2\mu_o}$$

The kinetic energy of a hydromagnetic wave may be derived as follows:

$j = j_0 \cos\omega t$

$$j_0 = \frac{2\pi b_0}{\mu_0 \lambda} \qquad j = \frac{2\pi b_0}{\mu_0 \lambda} \cos\omega t \; \sout{\sin\omega t}$$

$$\frac{dv}{dt} = \frac{jB}{\rho} = \frac{2\pi b_0}{\mu_0 \lambda} \frac{B}{\rho} \; \sout{\sin\omega t} \; \cos\omega t$$

$$v = + \frac{2\pi}{\lambda\omega} \frac{b_0 B}{\mu_0 \rho} \sin\omega t \; \sout{\cos\omega t} = + \frac{b_0}{\sqrt{\mu_0 \rho}} \sin\omega t \; \sout{\cos\omega t}$$

$$\text{since } V_{hm} = \frac{B}{\sqrt{\mu_0 \rho}}$$

$$\therefore \frac{1}{2}\rho v^2 \Big|_{av.} = \frac{1}{2} \cdot \frac{1}{2} \frac{\rho b_0^2}{\mu_0 \rho} = \frac{1}{4} \frac{b_0^2}{\mu_0}$$

$v_0 = \sqrt{2}\, v_{rms}$

This is the same as for the magnetic energy density. Therefore, the total energy density is

$$\left(\frac{b_0^2}{2\mu_0} = \right) \frac{b^2_{rms}}{\mu_0}$$

$\frac{1}{2}\rho v^2_{rms} = \frac{1}{4}\frac{b_0^2}{\mu_0}$

$v_{rms} = \frac{1}{\sqrt{2}} \frac{b_0}{\sqrt{\mu_0 \rho}} = \frac{1}{\sqrt{2}} v_0$

Also, note that $\left(v = \frac{b}{B} V_{hm}\right)$.

$v =$

$V_{hm} = B/\sqrt{\mu_0 \rho}$

$\therefore v/V_{hm} =$

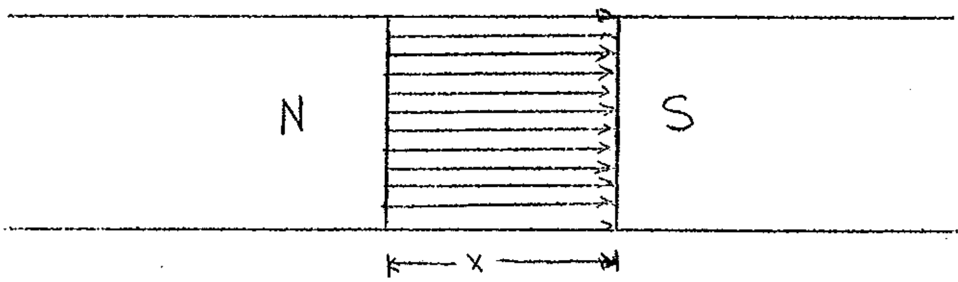
Notice that the radiation pressure, the excess pressure against the wall due to the wave, is of the order of the energy density of the wave.

<u>THE CONCEPT OF MAGNETIC TENSION</u> - We have, say, two pole pieces separated by a distance X and assume a uniform field between them. If the pole pieces have an area A, the total magnetic energy between the pole pieces is:

$$\mathcal{E} = \frac{B^2}{2\mu_0} A X$$

$$\frac{v_0}{V_{hm}} = \frac{b_0/\sqrt{\mu_0 \rho}}{B/\sqrt{\mu_0 \rho}}$$

$$\therefore v_0 = \frac{b_0}{B} V_{hm}$$



$$W = \frac{\mathcal{E}}{A} = \frac{B^2}{2\mu_0} \cdot \frac{Ax}{A} \ \frac{\text{joules}}{m^2}$$

Fig. 7-4

The energy per unit area of pole piece will increase linearly with x. In order to increase the energy, work must be done. Work is force times a distance. Thus, to solve for the force, we take the derivative of both sides.

$$F_A = -\frac{dW}{dx} \qquad F_A = -\frac{dW}{dx} = \left(-\frac{B^2}{2\mu_0}\right) \frac{\text{Newtons}}{M^2} = \frac{B^2}{2\mu_0} - T$$

$$\therefore T = \frac{B^2}{\mu_0}$$

The minus sign signifies that the pole pieces are being pulled together.

This is an interesting result. In other words, the magnetic field exerts an outward pressure perpendicular to the field lines, and an inward pressure along the field lines of a uniform magnetic field.

This term represents a pressure of magnitude $B^2/2\mu_0$ and the second term represents a tension of magnitude B^2/μ_0

David Cummings

10/9/63

SPACE SCIENCE 500

Lecture VIII

Today we will describe a geomagnetic field as it would appear in a vacuum with only currents existing in the earth's core producing the field, and, later, we will talk about perturbations due to trapped radiation, solar winds, etc.

The earth's field is generally quite dipolar. The deviations from dipole-like characteristics are of the order of 1%. There are a few localized regions where the deviations are much larger (i.e., magnetic anomalies). On closer examination, however, it is found that the magnetic center of the earth is displaced with respect to the spin-axis of the earth. Magnetic measurements that are worth very much have only been made for, say, the past 100 years. During this period of time, the geomagnetic field axis has been off-center and tilted with respect to the spin axis (see Fig. 8-1).

(Note: Show dipole pointed down).

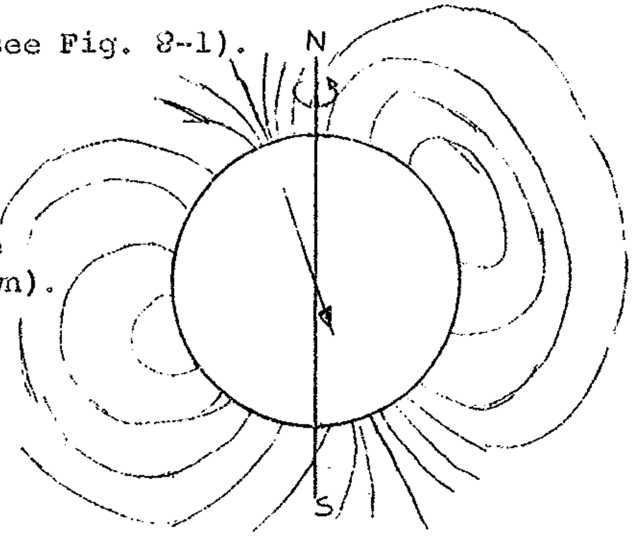


Fig. 8-1

At any point on the earth's surface, the geomagnetic field changes; these changes are called the <u>secular change</u>. The drift of the earth's main dipole moment due to this secular change are shown in Table I. I wouldn't take these numbers, especially the first two, very seriously, but the last two are really rather good, especially the yearly drift.

TABLE I

Year	Distance Off Spin Axis
1845	285 km
1855	305
1922	365
1945	338
1955	436

The secular change is not very important over a year or two, but, if you are doing accurate work, or over geologic time, it is important.

This outward shift of the dipole away from the earth shown in Table I might affect the Van Allen radiation and the system called "BL coordinates". Thus, if you are using 1955 geomagnetic data and you are working in 1963 or 1964, then have a look at the magnetic data if you find some inexplicable errors. Don't trust the stuff that comes out of the computer as gospel.

That these changes are significant over geologic time is evidenced from the magnetization of sediments that have hardened, pieces of pottery, etc., that show that the earth's field reverses about every 10,000 years or so. This is consistent with what is shown in Table I. This kind of motion in a hundred years may well lead to reversals every 10,000 years, or at least major changes. The earth's field is about 50,000 gammas; a secular change of only 10 gammas a year would reverse the earth's field in only 5,000 years. We will have a new harmonic analysis in 1965, I might add, so we will know very shortly what has happened to the geomagnetic field lately.

David Cummings

10/11/63

SPACE SCIENCE 500

Lecture IX

Last time we talked about the displacement of the magnetic center of the earth and how it varied with time and the secular change.

The secular variation was discovered in 1635 by means of data obtained as early as 1580, and yet today there is no idea at all as to what causes secular variation. The magnetic field of the earth is thought to be currents flowing in the earth's core, caused by convection-type currents--self-excited dynamo currents they are called--flowing in the earth's core. Since the earth core is conducting and the earth is rotating, there could be some differential rotation in the earth's core. Such a model can account for the earth's magnetic field. However, these convective motions have never been shown to be stable. Perhaps the 10,000 year lifetime I was indicating earlier for the earth's magnetic field is the characteristic instability period -- maybe the motions are not really stable, and it all goes to pieces every 10,000 years or so.

Briefly, it is thought the earth's field arises as follows: you start with a very weak field, say, the interplanetary magnetic field and a spinning planet. The planet very quickly wraps up the field and works on it. Energy is taken from the spin-motion to compress or increase the magnetic field. The result is a dipole field.

For example, Jupiter, which is spinning very rapidly, has a strong magnetic field. The spin-axis of Jupiter is observed by the motion of the cloud-markings. Also, a radiation belt, like a Van Allen radiation belt, is detected around Jupiter. The magnetic field is inclined at 9° to the spin-axis of the planet. The earth's magnetic moment is inclined more like 11° to the spin axis. What strikes most people about these two observations is the apparent similarity in the inclination of the two dipole moments. And, if Mariner passes as close to Venus as claimed, the magnetometer results imply that Venus doesn't have a magnetic field that is even a tenth as big as the earth's moment. Radar observations show that Venus is either rotating very, very slowly or not at all. These observations fit the theory that you need a spinning planet with a spinning core with convective and coriolis forces to work up a magnetic moment starting with a weak interplanetary field.

Let's go on to the spherical harmonic analysis which, in one sense, is a pretty dull, tedious business, and yet is very important and produces interesting results. First, there are some magnetic nomenclature that we should all commit to memory (see page 120 of the Satellite Environment Handbook). The total magnetic intensity is always given as F and the inclination or magnetic dip, I, is the angle between the horizontal and F. The dip angle is 90^{o} at the magnetic poles. At the magnetic equator, the dip is zero. The dip is positive in the northern hemisphere -- that is, when the north end of the compass points down, the dip is positive. And, by the way, the north pole of the earth is really a south magnetic pole. This has to be so because the north end of a magnet will point north and, therefore, it is being attracted by a south magnetic pole. The oldtimers, you will see, always draw the magnetic moment of the earth pointing down, and it might be a good way for us to do it, too.

The declination, D, is the angle between the true north and the direction the compass points. Declination is positive when magnetic north is to the east of true north.

H is the horizontal component of intensity of the earth's magnetic field. Magnetic storms show most clearly in the H component, so it is used most often.

$$H = F \cos I > 0 .$$

V is the vertical intensity and it has the same sign as I. Thus

$$V = F \sin I .$$

F is hardly ever used; the old instruments couldn't measure F and it is just not the fashion to measure the total field. The reason H shows the main magnetic storm variations is because the earth is a fairly good conductor. For periods like a day, the skin depth of the earth is not very deep -- it is just a few hundred kilometers. So, it is hard to change the vertical component because of the conductivity of the earth (you have a frozen-in flux essentially) and the vertical component on the earth can't change very much. But you can squeeze the horizontal component against the earth and compress it or release it, i.e., H can be more easily changed than V. The vertical component hardly shows any change at all during a magnetic storm.

Occasionally, X, Y, and Z are used, where X is the north component of H. Y is the east-pointing component, and Z is the same as V, the vertical component. Thus

$$X = H \cos D, \quad Y = H \sin D, \quad Z = V .$$

For the spherical harmonic analysis of the geomagnetic field, X, Y, and Z are used -- that is why I brought them up.

SPHERICAL HARMONIC ANALYSIS

There is a very good discussion of this spherical harmonic analysis starting on page 639 of chapter 18, Vol. 2 of Chapman and Bartels.

Given a curve over some interval, we can represent it as a series of sines and cosines. Likewise, given a sphere where we know a function over the entire surface of the sphere, we can represent this function by a series of Lengendre polynomials. This representation of the earth's magnetic field by such a series is known as the spherical harmonic analysis.

First of all, we will assume that the earth's magnetic field can be represented as the negative gradient of a scalar potential, Φ_M, so that

$$X = \frac{1}{r}\frac{\partial \Phi_m}{\partial \theta}, \quad Y = -\frac{1}{r\sin\theta}\frac{\partial \Phi_m}{\partial \lambda}, \quad Z = \frac{\partial \Phi_m}{\partial r}$$

where r, θ, λ are the usual spherical coordinates.

X, Y, and Z are measured on the earth's surface -- that is, the primary data. We assume that the earth's magnetic field is made up of an internal and external contribution. I have said before that the earth's magnetic field comes from currents inside the earth's core. But for the spherical harmonic analysis, you don't have to assume that. As a matter of fact, you can use the spherical harmonic analysis to prove that the major part of the earth's magnetic field comes from the earth's core.

$$\Phi_{m_{int}} = R_E \sum_{n=1}^{\infty} \left(\frac{R_E}{r}\right)^{n+1} T_{n_{int}},$$

$$\Phi_{m_{ext}} = R_E \sum_{n=1}^{\infty} \left(\frac{r}{R_E}\right)^{n+1} T_{n_{ext}}$$

$\hat{\Phi}_{Mint}$ is that part of the potential due to internal sources; $\hat{\Phi}_{Mext}$ is that part due to external sources. The quantity T_n is

$$T_n = \sum_{m=0}^{n} (g_n^m \cos m\lambda + h_n^m \sin m\lambda) P_n^m(\cos\theta)$$

where $P_n^m(\cos\theta)$ is the associated Legendre polynomial. The quantities g_n^m and h_n^m are called Gauss coefficients in honor of Carl Frederick Gauss who was the first man to realize you could use this machinery to tell whether the magnetic field of the earth comes from inside or outside the earth. He pointed this out in a published paper. After quite a few years, there was a very crude world survey; he was able to show that most of the earth's magnetic field came from internal sources.

As a matter of fact, my understanding is that you only need the Z component of the earth's field to separate the internal from the external contributions.

All the work is done in geographic coordinates. Look at the Tables of Gaussian (Schmidt) Coefficients (1955) - Table 9-1 very quickly and we will learn something almost immediately. The British, U.S., and U.S.S.R. all use the same primary data. And yet, you will notice the numbers are different. This is because before they use the data, they interpret it. Notice that the coefficients agree to about 1%. That is because the data is good to about 1% and the interpretations don't really help any.

One interesting point is the first term. If you multiply by 10^{-4} like it says at the top, the term n = 1, m = 0 is .3055 for the British, .3054 for the U.S., and .3051 for the U.S.S.R. That gives the magnetic field in Gauss at the magnetic equator of the earth for the first dipole term. It is the equatorial strength of the main dipole of the earth's magnetic field.

TABLE 9-1. Spherical Harmonic Analyses of the Geomagnetic Field for 1955 Basis, British, U.S., and U.S.S.R. World Charts. Gaussian (Schmidt) Coefficients, g_n^m, h_n^m, in units of 10^{-4} cgs.
(After Word Magnetic Survey Instruction Manual, E. H. Vestine, Ed.)

		g_n^m			h_n^m	
n,m	British	U.S.	U.S.S.R.	British	U.S.	U.S.S.R.
1,0	-3055	-3054	-3051			
2,0	- 152	- 147	- 141			
3,0	118	117	112			
4,0	95	87	97			
5,0	- 27	- 24	- 33			
6,0	10	2	7			
1,1	- 227	- 210	- 202	590	585	584
2,1	303	307	299	190	-185	-187
3,1	- 191	- 170	- 174	- 45	- 59	- 56
4,1	80	64	78	15	18	11
5,1	32	40	33	2	10	10
6,1	5	12	8	- 2	- 6	- 7
2,2	158	145	168	24	49	38
3,2	126	127	124	29	30	26
4,2	58	47	57	- 31	- 24	- 31
5,2	20	21	16	10	10	12
6,2	2	- 2	3	11	16	14
3,3	91	86	81	- 9	- 3	- 8
4,3	- 38	- 44	- 36	- 4	- 7	- 5
5,3	- 4	- 4	- 9	- 5	- 1	- 4
6,3	- 24	- 26	- 26	0	0	- 1
4,4	31	29	32	- 17	- 13	- 13
5,4	- 15	- 15	- 14	- 14	- 18	- 14
6,4	- 3	- 7	- 3	- 1	0	0
5,5	- 7	- 4	- 6	9	8	1
6,5	0	3	1	- 3	- 4	- 1
6,6	- 11	- 10	- 9	- 1	- 4	- 3

It should be noted that there is little difference between these 3 sets of coefficients. The differences are ∿ 1% which is the order of the uncertainty in the primary geomagnetic data.

David Cummings

10/14/63

SPACE SCIENCE 500
Lecture X

Last time we noted that the first term of the Gaussian coefficients gives the magnetic field at the geomagnetic equator. We noticed that we could fit the geomagnetic field with an internal and external potential. Within experimental error (~1%) the external potential describes the earth's field, and therefore it is found at least 99% of the earth's field is due to internal currents--less than 1% comes from the outside. Sometimes you see smaller numbers for the external contribution; but the primary data isn't good enough to really limit the external contribution to any less than 1%. Only the Z component or the vertical component of the earth's field is needed to separate the internal from the external contribution.

Two more points and we are through with spherical harmonic analysis. This first point involves the assumption that B is given by a negative gradient of a scalar potential. If we take the curl of $\bar{B}$, we find that

$$\bar{\nabla} \times \bar{B} = -\bar{\nabla} \times (\bar{\nabla} \Phi_M), \text{since} \quad \bar{B} = -\vec{\nabla} \Phi_M$$

and we get minus the curl of a gradient, which is identically zero. So, by assuming $\bar{B} = -\vec{\nabla}\Phi_M$, we are implying that the curl of $\bar{B}$ is zero, and this means that the conduction current and the displacement current in the region where we are taking the data are zero. If this were not so, when the spherical harmonic analyses were fitted to the field, there would be a residue (it is called) to the scalar potential. There are such currents (e.g., thunderstorm currents, lightening flashes, and auroral electrical currents), but they are so weak that they cannot be detected by this method.

Future spherical harmonic analyses will probably improve in accuracy over the past ones by a factor of 3 to 5. It is going to be quite difficult to improve much more because of magnetic noise such as diurnal variations that cause big changes in the field. Also, it is hard to make these measurements on ships to better than, say, 50 gammas. The present spherical harmonic analysis is good to about 300 gammas.

The final point is how the magnetic center of the earth is to be displaced. The Van Allen radiation shows the displacement. The bottom of the Van Allen radiation belt starts at about 1,000 kilometers altitude over the Pacific, and is 200 or even 100 km. over the Atlantic side. Thus, the dipole is displaced towards the Pacific side. This displacement can be gotten out of spherical harmonic analysis as a first order effect by looking at all the terms and choosing a new origin such that the first few coefficients vanish. In Chapter 18 of Chapman and Bartels, page 639, you will find a formula given for setting these various coefficients to zero. It will locate, in a very simple way, the coordinates of the magnetic center at which these first few coefficients of the spherical harmonic analysis vanish. The method was first pointed out by Kelvin.

The dipole term, interestingly enough, doesn't change. The dipole is just translated from the center to the new location. The new magnetic center will agree fairly well with the height variation of the Van Allen radiation belt with longitude.

In conclusion, the spherical harmonic analysis can (1) separate internal from external contributions to the earth's magnetic field, and (2) allow the calculation of $\bar{B}$ as a function of r, θ and λ -- that is, you can calculate $\bar{B}$ any place above the earth by just using these coefficients and plugging into the formulas given in a previous lecture (there are machine programs for doing this).

However, remember that the spherical harmonic analysis found no external contributions to the earth's field, and we know that there are some. For example, the Van Allen radiation belt or the solar wind blowing against the earth's magnetic field provide a weak, external contribution. However, the spherical harmonic analysis is not sufficiently delicate to detect these, so when you use the spherical harmonic analysis to predict the earth's field, it is good near the earth, and out a few earth radii, say, 4 earth radii, but beyond there, it begins to deviate very badly. (3) The spherical harmonic method can find the magnetic center of the earth. There is a good description of exactly where the magnetic center was in 1955 on page 132 of the Satellite Environment Handbook.

The equation for a dipole field line is

$$r = L \cos^2 \lambda_m$$

where λ_m is the magnetic latitude; L is the geocentric distance to where the line crosses the magnetic equator and r is the geocentric distance to a point on the line having latitude λ_m.

HOMEWORK PROBLEMS

1. (a) What is a magnetic line of force?

   (b) Derive:

$$r = L \cos^2 \lambda_m .$$

2. What is the total magnetic energy external to the earth? (Assume a perfect dipole with an equatorial field strength .305 gauss and a radius 6.37×10^3 km.).

3. Show that the integral $\int \bar{B} \cdot \bar{H} dV$ over all space for a uniformly magnetized sphere vanishes. It doesn't have to be a sphere, but I think you will find it a lot easier if you use a sphere.

David Cummings

10/21/63

SPACE SCIENCE 500

Lecture XI

We just have one more thing to do with the static dipole field and then we can go into magnetic coordinate systems.

In a previous lecture, I derived the magnetic tension for a straight uniform magnetic field, the force per unit area (which was the negative derivative of the energy per unit volume).

$$F = -\frac{\partial W}{\partial x} = -\frac{B^2}{2\mu_o} \quad \text{where} \quad W = \frac{B^2}{2\mu_o} x$$

but F is made up of an isotropic pressure $+\frac{B^2}{2\mu_o}$ and a tension T.

$$\therefore \quad F = \frac{B^2}{2\mu_o} - T = -\frac{B^2}{2\mu_o} \Rightarrow T = \frac{B^2}{\mu_o} \quad .$$

A more exact treatment of magnetic tension, using the Maxwell stress tensor, is given in "Classical Electricity and Magnetism" by Panofsky and Phillips, Sect. 6-5.

If we curve the field line as shown in Fig. 11-1,

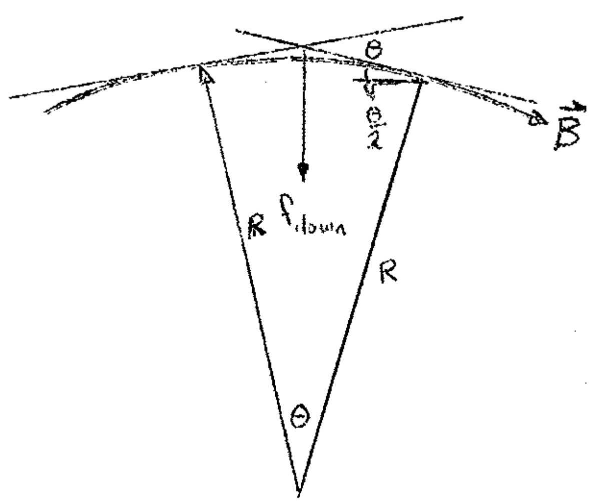


Fig. 11-1

there is a downward force equal to the tension times θ/2 for one side, and I also get an equal force from the other side -- so I just double it.

For a given tube of flux (the one we have just considered), the force per unit length is

$$\frac{f}{\ell} = \frac{T\theta}{R\theta} = \frac{T}{R} = \frac{B^2}{\mu_o R}$$

This force per unit length has to be balanced by something, and what it is balanced by is the perpendicular gradient of the magnetic pressure. Thus,

$$\frac{2}{R}\left(\frac{B^2}{2\mu_o}\right) = \nabla_\perp\left(\frac{B^2}{2\mu_o}\right) \rightarrow \frac{\nabla_\perp B}{B} = \frac{1}{R}$$

$$\nabla_\perp\left(\frac{B^2}{2\mu_o}\right) = \frac{T}{R} \rightarrow \frac{B\nabla_\perp B}{\mu_o} = \frac{B^2}{\mu_o R}$$

or $$\frac{\nabla_\perp B}{B} = \frac{1}{R}$$

This is an important result. Let's discuss why it is important. For one thing, if the radius of curvature is infinite, that is, if the lines of force are straight, there can be no perpendicular gradient of B.

In current-free regions, any time the field lines are curved, you must have a gradient of B, and any time you have a gradient B, you know the field lines are curved.

Let's do this one more time in a more rigorous manner. From Maxwell's equation we know that in a current-free region, $\bar{\nabla} \times \bar{B} = 0$. Now let's draw two field lines,

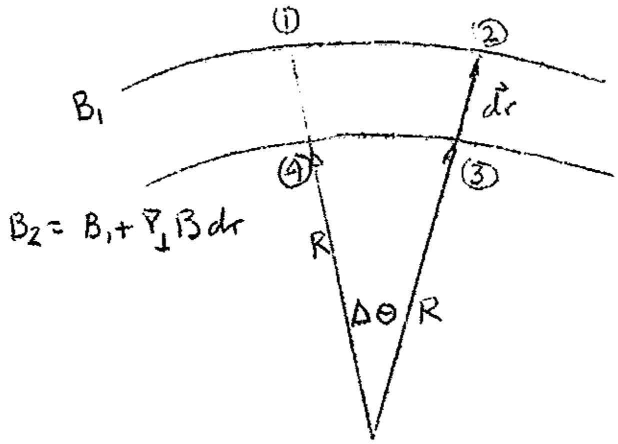


Fig. 11-2

We have a perpendicular gradient of B; the field is increasing toward the origin. We take a line integral around the path 1, 2, 3, and 4. Ignoring terms of second order,

$$\int \vec{B}\cdot d\vec{\ell} = 0 = B_1(R+dr)d\theta - (B_1 + \nabla_\perp B_1 dr)Rd\theta$$

$$\therefore \quad B = (\nabla_\perp B)R \rightarrow \frac{\nabla_\perp B}{B} = \frac{1}{R}$$

This derivation is quite general. It could be done a little more elegantly, but it wouldn't show anything more. There are no assumptions except that of no current. What these proofs are supposed to do is to show you what it is that is balanced against what; why, when you get a curved field line, you need a pressure gradient or a gradient of B. This is why the field of a dipole doesn't explode -- the pressure gradient is pushing out, and the tension is holding it together.

I am going to go on now to a new topic -- magnetic coordinate systems.

(1) The first coordinate system that you use to refer everything else to are ordinary geographic coordinates. As we discussed before, the earth's magnetic dipole is tilted about 11° in the

geographic axis and displaced a few hundred kilometers from the earth's center. It is pretty obvious that anything that is controlled magnetically will not be described well by geographic coordinates.

(2) The next one you might use is geomagnetic coordinates. Here make a note to look on page 130 of the <u>Satellite Environment Handbook</u> where there is a map of geomagnetic coordinates. Geomagnetic coordinates are just the coordinate system defined by the single centered dipole that best fits the earth's magnetic field. The single-centered dipole is on the earth's spin-axis. Remember the dipole is tilted 11^{o} and displaced a few hundred kilometers; for the geomagnetic coordinate system, the displacement is ignored. You assume the earth's magnetic field is represented by a tilted but centered dipole, and you get a real dandy coordinate system -- poles, an equator, etc. But, of course, you realize that this is very artificial and can't be very good.

There was quite a to-do in about 1956 when Simpson of Chicago measured the cosmic ray equator and found it displaced from the geomagnetic equator.

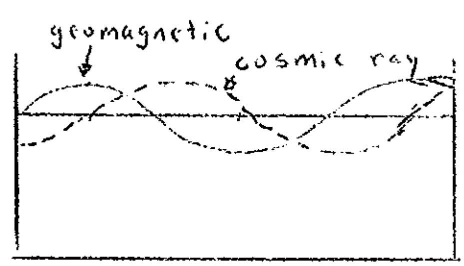


Fig. 11-3

The cosmic ray equator turned out to be shifted about 45^{o} from the geomagnetic equator. From what I have just told you, none of you should be surprised -- in fact, you might be surprised that it fit that well. The geomagnetic coordinate system is only a little bit better than the geographic coordinate system. None of you should be surprised, although you will find geomagnetics used in all the old literature and is still being used; it is really not worth much.

(3) A magnetic coordinate system that you can use very easily, and that is fairly good, is called the "dip latitude coordinate system." The magnetic dip is the inclination of the field line from the horizontal. We define λ_m, the dip latitude, as arctan ($\frac{1}{2}$ tan I) where I is the dip angle.

This formula is exactly true for a perfect dipole, and it is not true for a deformed dipole like the earth's field. But it is close enough to give you a very good coordinate system. If we go back to this magnetic cosmic ray equator and geomagnetic equator and draw the dip equator -- that is, the equator where the dip angle here is zero -- it gives an excellent fit within experimental error. The aurorae occur along a line of constant dip. Most geomagnetic phenomena do.

HOMEWORK PROBLEM

Assume a perfect dipole and prove that

$$\lambda_m = \tan^{-1} [\tfrac{1}{2} \tan I]$$

(4) The best system to use is one that uses the properties of the particles themselves. If you had a system that actually described how particles moved in the field and described latitudes in terms of these particle motions, you would have quite a good useful system because the reason you want magnetic coordinate systems is to describe Van Allen radiation, aurorae, cosmic rays and things like that.

The system that is now being used is an integral invariant coordinate system called the "B, L coordinate system." To define this B, L coordinate system, we will first need to define the invariants and how the particles move in a magnetic field.

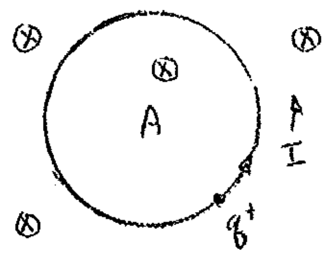


Fig. 11-4

First, we will show that the magnetic moment, $\vec{u}$, of a charged particle moving in a magnetic field is a constant. The magnetic moment is generally defined as the current around a loop times the area of the loop. If we have a single particle with charge e going around in a circle (for instance, motion in a uniform magnetic field), the current is equal to the number of times per second it goes around (which is $\omega_c/2\pi$). The area is just πa_c^2 where a_c is the cyclotron radius. If we multiply these two together,

$$u = \frac{e\omega_c a_c^2}{2}$$

But $$a_c = \frac{m v_\perp}{eB} \text{ and } \omega_c = \frac{eB}{m}$$

Then $$u = \frac{\frac{1}{2}mW_\perp^2}{B} = \frac{(KE)_\perp}{B}$$

The magnetic moment of a particle in a magnetic field is the component of energy perpendicular to the magnetic field divided by the magnetic field.

We can prove that u is a constant for certain cases. There are two proofs; one of them is originally by Alfven and the other is the one used by Spitzer. You will see that it isn't really a very rigorous proof because it assumes what is to be proved. Of course, it works out right! But, it is a good proof to go through because it illustrates the mechanism by which the moment is conserved.

The particle going around in a circle has the above magnetic moment. If we change $\overline{B}$, we know we get a curl of $\overline{E}$. This is independent of whether a particle is there or not, and independent of whether you say there is a conductor there that shorts out the $\overline{E}$ field.

The electric field, so produced, will accelerate the particle. The EMF around the loop is

$$\text{EMF} = \int \overline{E} \cdot d\overline{s} = - \int \frac{d\overline{B}}{dt} \cdot d\overline{s}$$

The electric field will accelerate the particle so that the change in kinetic energy per unit time is given by the current times the EMF around the loop. Thus,

$$\frac{d}{dt}\left(\frac{1}{2} m w_{\perp}^2\right) = \left(\frac{\omega_c}{2\pi} e\right)\left(\pi a_c^2 \frac{dB}{dt}\right) = u \frac{dB}{dt}$$

But $$uB = \frac{1}{2} m w_{\perp}^2$$

so that $\frac{d}{dt}(uB) = u\frac{dB}{dt}$, which implies that u = constant! Q.E D.

However, note that this proof requires that ω_c and a_c^2 (or at least their product) be constant. If $\omega_c a_c^2$ = const., then since $\omega_c = \frac{e}{m} B$ and $a_c = \frac{m w_{\perp}}{eB}$, $\omega_c a_c^2 = \frac{m w_{\perp}^2}{eB}$ which is $\frac{2u}{e}$ = const $\rightarrow u$ = const.

Also, in this proof $\int \frac{d\overline{B}}{dt} \cdot d\overline{s} = \pi a_c^2 \frac{dB}{dt}$. But if $u = \frac{\frac{1}{2} m w_{\perp}^2}{B}$ = const.,

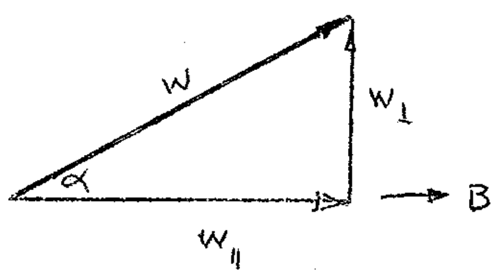
then $$u \cdot \frac{2\pi m}{e^2 B} B = \frac{\pi m^2 w_{\perp}^2}{e^2 B^2} B = \pi a_c^2 B = \text{const.}$$

That is, if u = const., then flux through loop $\pi a_c^2 B$ can not change -- hence, no EMF and no acceleration.

But it must be quite obvious to you that if the particle is going around in a certain sized circle and then the field is changed slowly, at first the size of the circle wouldn't change, and then it would begin following the field change, lagging a little bit behind. And then, when the field stopped changing, it would overshoot, and the size of the circle would oscillate a little bit about the equilibrium position. It's a complicated dynamical problem.

Before I give the rigorous proof, let me go through a couple of small points for you. Suppose a particle is moving with a total velocity w and $w_\perp$ perpendicular to B, and a $w_\parallel$ $\parallel$ to B as shown below. Then the magnetic moment is

$$u = \frac{\frac{1}{2} m w_\perp^2}{B} = \frac{\frac{1}{2} m w^2 \sin^2\alpha}{B} = \text{constant}$$



But at $\alpha = \pi/2$, $u = \frac{\frac{1}{2} m w^2}{B_m} = \frac{(KE)_{tot.}}{B_m}$

where B_m is the mirror field at $\alpha = 90°$.

Thus $\frac{\sin^2\alpha}{B}$ = constant

α is called the pitch angle. The pitch angle is the angle between the instantaneous velocity vector of a particle and the local magnetic field. As $\bar{B}$ gets stronger, α has to become bigger in order for u to remain constant. The limit is where α is $\pi/2$, then this is the deepest the particle can penetrate into a field that is growing stronger with distance; that is, the maximum field this particle can ever reach is given by the total energy divided by the magnetic moment.

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10/23/63

SPACE SCIENCE 500

Lecture XII

Last time we went through a derivation that showed the magnetic moment u was constant. Let me talk about that derivation a little more -- I said that the derivation assumed what was to be proved. I wrote the derivative of the kinetic energy with respect to time

$$\frac{d}{dt}(KE_{\perp}) = \frac{d}{dt}\left(\frac{1}{2}mw_{\perp}^{2}\right) = \left(\frac{\omega_c}{2\pi}e\right)\pi a_c^{2}\,\frac{dB}{dt}$$

This derivation assumes that $\omega_c a_c^2$ is a constant. It doesn't assume that both ω_c and a_c are constants; it just assumes that the product of the two is a constant.

Let us work on this a bit. We write $\omega_c = eB/m$ and $a_c^2 = w_{\perp}^2 \left(\frac{e}{m}\right)^2 B^2$. The product $\omega_c a_c^2$ is $\frac{mw_{\perp}^2}{eB}$ -- a constant. Since $u = \frac{\frac{1}{2}mw_{\perp}^2}{B}$,

$\frac{mw_{\perp}^2}{eB} = \frac{2u}{e}$; therefore, by assuming $\omega_c a_c^2$ = constant, we have assumed u = constant. This derivation, as presented by Spitzer and Alfven, serves just to illustrate the mechanism rather than to rigorously show the principle; it shows how the electric field from the changing magnetic field accelerates the particle.

Finally, using $\omega_c a_c^2$ as a constant,

$$\omega_c a_c^2 = \frac{eB}{m}a_c^2 = \frac{e}{\pi m}B\pi a_c^2 = \left(\frac{e}{\pi m}\right)\Phi\ .$$

Thus, the magnetic flux Φ through the particle trajectory is a constant. As the particle moves in stronger or weaker fields, the cyclotron radius always adjusts so that the flux through the particle trajectory is a constant if the magnetic moment is constant. This is an important result that we can use at various times.

One other way you might think about it -- the particle behaves a little bit like a superconducting ring, but since it doesn't have enough energy to push the field around, it adjusts its own size so the flux through the superconducting ring doesn't change.

Let us define α as the angle between the tangent to the magnetic field and the tangent to the total velocity vector; even though they both may be curving, you can take an instantaneous value. α is usually called the pitch angle. We may now rewrite u in terms of α:

$$u = \frac{\frac{1}{2}mw_{\perp}^2}{B} = \frac{\frac{1}{2}mw^2\sin^2\alpha}{B} = \text{const.}$$

Since $\frac{1}{2}mw^2$, the total particle energy, is constant, $\frac{\sin^2\alpha}{B} = \text{const.}$ As B increases, α will increase until it becomes 90°. Then the particle can't go any farther and will bounce back. The maximum field is called the mirror field Bm.

We can see this mechanistically if we look at some field lines and look at a particle moving in a converging field (see Figure 12-1).

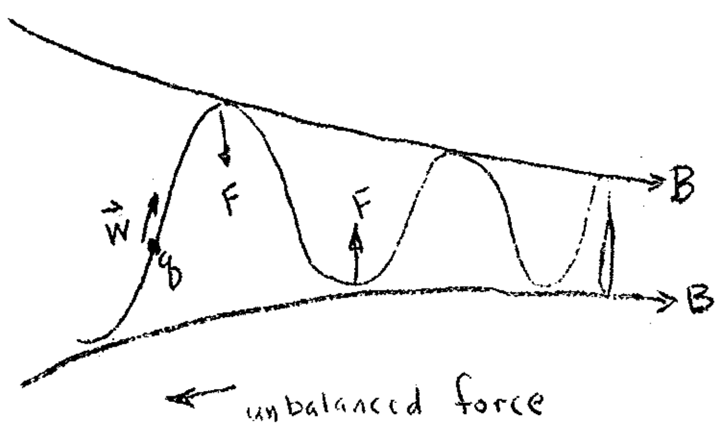

Fig 12-1

The force on the particle is $\overline{F} = e\overline{w} \times \overline{B}$. The force on the particle has a component that accelerates the particle toward weaker fields.

We can write down the acceleration since we know the particle has a constant magnetic moment u, and there is a gradient in the field. If you take a magnetic moment or a little dipole (and the dipole moment is pointed antiparallel to the field), then there is a force on this dipole, $-u\nabla_{\parallel}B$, where $\nabla_{\parallel}B$ is the component of ∇B parallel to B. Thus,

$$\overline{F} = -u\overline{\nabla_{\parallel}B} = m\frac{d\overline{w}_{\parallel}}{dt}$$

Since this acceleration is always in the same direction, the particle will eventually slow down and stop, and then go back along B. That is another way to look at the mirroring process. In many cases, it is useful to think of particles as being replaced by charged dipoles. The dipole moment has the charge of the particle on it, and this charged dipole will exhibit all the properties of the spiralling microscopic motion, as long as the guiding center approximation holds.

The following is a rigorous proof of the constancy of the magnetic moment u. Let's take a region of space in which the field lines are straight -- in other words, if we extended them in to an origin or intersection, then all the field lines would come to a point. We are considering the region of space where the field is $1/R^2$ field, the field radiated from a monopole. A $1/R^2$ dependence is the only one in which all the field lines are straight, and the field is still a function of position. For other dependences, the magnetic moment is not actually a constant. The magnetic moment will, in general, oscillate about an average position. In fact, it may oscillate so radically that the magnetic moment breaks down.

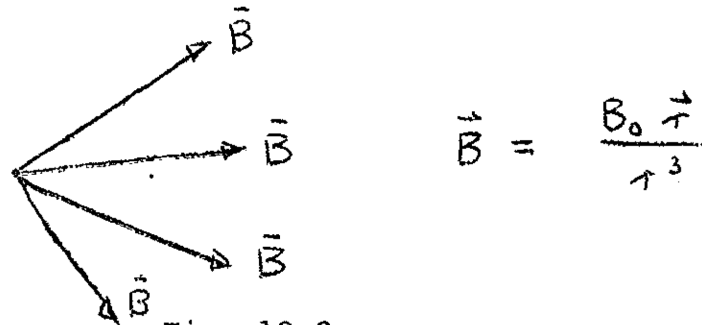

Fig. 12-2

The time rate of change of momentum of a charged particle is

$$\frac{d\bar{p}}{dt} = e\bar{w} \times \bar{B} = e\bar{w} \times \bar{r} \left(\frac{B_o}{r^3}\right).$$

Crossing both sides with $\vec{r}$,

$$\vec{r} \times \frac{d\bar{p}}{dt} = e\bar{r} \times \bar{w} \times \bar{r} \left(\frac{B_o}{r^3}\right) = \frac{eB_o}{r^3} [\bar{w}r^2 - \bar{r}(\bar{w}\cdot\bar{r})].$$

Now, take the time derivative of $\bar{r} \times \bar{p}$; I get two terms:

$$\frac{d}{dt}(\bar{r} \times \bar{p}) = \bar{w} \times \bar{p} + \bar{r} \times \frac{d\vec{p}}{dt} .$$

But the momentum $\bar{p}$ and $\bar{w}$ are parallel since the momentum is just $\bar{w}$ times the mass, and thus $\bar{w} \times \bar{p} = 0$.

$$\therefore \quad \vec{r} \times \frac{d\bar{p}}{dt} = \frac{d}{dt} (\bar{r} \times \bar{p}).$$

Substituting,

$$\frac{d}{dt}(\bar{r} \times \bar{p}) = \frac{eB_o}{r^3} [\bar{w}r^2 - \bar{r}(\bar{w}\cdot\bar{r})] = eB_o\left[\frac{\bar{w}}{r} - \frac{\bar{r}(\bar{w}\cdot\bar{r})}{r^3}\right] .$$

Note that: $\frac{d}{dt}\left(\frac{\bar{r}}{r}\right) = \frac{\bar{w}}{r} + \bar{r} \frac{d}{dt}\left(\frac{1}{r}\right) = \frac{\bar{w}}{r} - \frac{\bar{r}}{r^2} \frac{\bar{w}\cdot\bar{r}}{r}$

$$\frac{d}{dt}(\bar{r}) = \frac{(\bar{w}\cdot\bar{r})}{r} ,$$

so that we can write, $\frac{d}{dt}(\bar{r} \times \bar{p}) = eB_o \frac{d}{dt} (\frac{\bar{r}}{r})$.

Now integrating with respect to time, we get $\bar{r} \times \bar{p} = eB_o \frac{\bar{r}}{r} + \bar{C}$ or:

$\bar{r} \times \bar{p} - eB_o \frac{\bar{r}}{r} = \bar{C}$ and taking magnitudes,

$$(\bar{r} \times \bar{p} - eB_o \frac{\bar{r}}{r})^2 = \bar{C}\cdot\bar{C} = C^2 = |\bar{r} \times \bar{p}|^2 + e^2B_o^2 \frac{|\bar{r}|^2}{r^2} - \frac{2eB_o\bar{r} \times \bar{p}\cdot\bar{r}}{r}$$

$\rightarrow |\bar{r} \times \bar{p}|^2 = C^2-e^2B_o^2$. But $|\bar{r} \times \bar{p}| = rp \sin \alpha$, where α is the angle now between the radius vector (also the direction of the magnetic field), and $\bar{p}$, or, in other words α is the "pitch angle." And $\bar{p}$ is the particle mass times the velocity perpendicular to $\bar{R}$.

$$\therefore r^2p^2\sin^2\alpha = C^2-e^2B_o^2 .$$

But $B = B_o/r^2 \rightarrow p^2 \frac{\sin^2\alpha}{B} = \frac{(C^2-e^2B_o^2)}{B_o}$.

The particle momentum times $\sin^2\alpha$ divided by B is a constant. Since the particle momentum (the total momentum, the total velocity) is also a constant, we are left with $\sin^2\alpha/B$ = const. From this result, as demonstrated previously, we can show that

$$\frac{\frac{1}{2}mw_1^2}{B} = u = \text{const.}$$

David Cummings

10/25/63

SPACE SCIENCE 500

Lecture XIII

Consider a particle in a magnetic field. The particle will have a magnetic moment u where

$$u = \frac{1}{2}\frac{mw_{\perp}^2}{B}.$$

There will be a magnetic force on the particle $\overline{F}_M$ where $\overline{F}_M = -\overline{u}\cdot\nabla_{\perp}\overline{B}$. $\nabla_{\perp}B$ is the component of ∇B perpendicular to B. F_M is the force perpendicular to the magnetic field. Any time you apply a force to a charged particle in a magnetic field, it will move perpendicularly to the magnetic field and the applied force. So, if I apply a force, say, an electric field, the particle drifts perpendicularly to both $\vec{E}$ and $\vec{B}$. Thus, the particle tends to move perpendicularly to the magnetic field and the applied force, but, not immediately. The first motion is in the direction of the force. Then, as the particle starts moving in the magnetic field, it curves in the magnetic field. Thus, in general,

$$u\nabla_{\perp}B = e\vec{v}_D \times \overline{B}$$

Since I have just taken the perpendicular gradient of the magnetic force, $\vec{v}_D$ and $\vec{B}$ are perpendicular. Remembering that $|\vec{\nabla B}/B|$ is $1/R$, we write

$$v_{DM} = \frac{u|\nabla_{\perp}B|}{eB} = u/eR = \frac{1}{2}\frac{mw_{\perp}^2}{eBR}.$$

There is another motion that produces another force. The particle, besides having a $w_{\perp}$ to give it a magnetic moment, also has a $w_{\parallel}$ as it moves along the field line. There is a centrifugal force,

$$F_C = mw_{\parallel}^2/R_C$$

This centrifugal force is perpendicular to the magnetic field.

Thus, $mw_{\parallel}^2/R_C = ev_{DC}B$,

$$v_{DC} = mw_{\parallel}^2/eBR_C \ .$$

So, the total drift velocity in a curved magnetic field is

$$V_D = v_{DM} + v_{DC} = \frac{m}{eBR_C}\left(\frac{w_{\perp}^2}{2} + w_{\parallel}^2\right) \ .$$

They both add, since they both have the same sign. For a given velocity squared, the centrifugal force is twice as effective. If the two velocities are equal, i.e., the particle has a 45° pitch angle, the parallel velocity component gives twice the drift contribution that the perpendicular one does.

Now, for a dipole field. The earth has a magnetic moment and the particle spiralling back and forth, has a magnetic moment that will be parallel to the earth's magnetic moment. These two dipole moments repel each other to give an outward force. Thus, the particle will drift around the earth. In addition, it will bounce back and forth giving centrifugal force also outward -- both forces are out so their contributions to the drift must add.

As a homework problem, what is the direction of drift in the earth's magnetic field for (a) electrons, and (b) positive ions? Draw a picture looking down on top of the north pole. That is, in the earth's magnetic field, how large an electric field would you need to set up at some point in space (pick your own parameter)?

For the second problem, calculate how large an electric field would you have to set up to get a drift velocity in the earth's field equal to that for a 50 kv electron or 1 Mev proton at 2 earth radii geocentric distance.

There is another way to look at the gradient drift. A particle in a uniform field will go around in a circle. As shown in Fig. 13-1, if

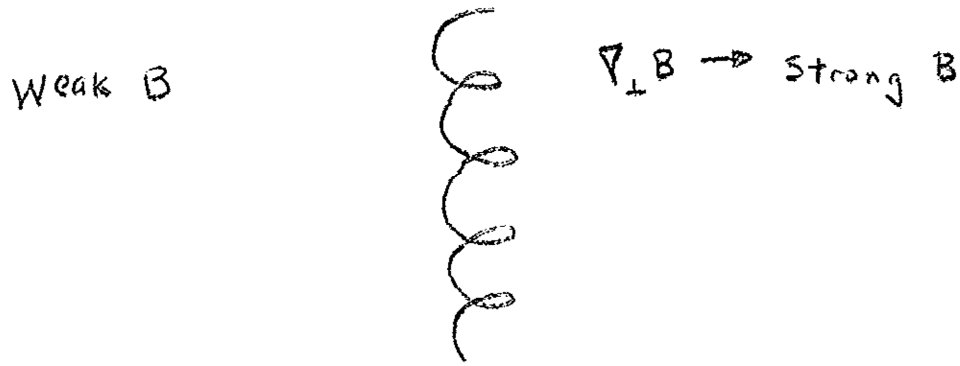

Fig. 13-1

there is a field gradient, the cyclotron radius will be less in the strong field than in the weak field. A gradient drift results as sketched. I think the dipole-dipole repulsion formulation is more powerful because it shows you what forces are acting on a particle in a more useful way. When we get to magnetic storm theory, the effect of the belt of trapped particle in the earth's magnetic field can be described better by thinking of charged dipoles being repelled by the earth's magnetic dipole.

And a final interesting point. Suppose you remove the charge from an equivalent dipole so that it moves away from the earth. What do you suppose its kinetic energy would be at infinity? Well, it would be $\frac{1}{2} mw_{\perp}^2$ since the potential energy of the dipole in the earth's field is

$$uB = \frac{\frac{1}{2}mw_{\perp}^2}{B} B = \frac{1}{2} mw_{\perp}^2 .$$

This potential energy must equal the kinetic energy at infinity.

Now consider particles moving in the earth's magnetic field. If the earth's field were axially symmetrical, you could predict from symmetry just where a particle will be as it drifts around. Let's say the earth's fields were as shown in Fig. 13-2.

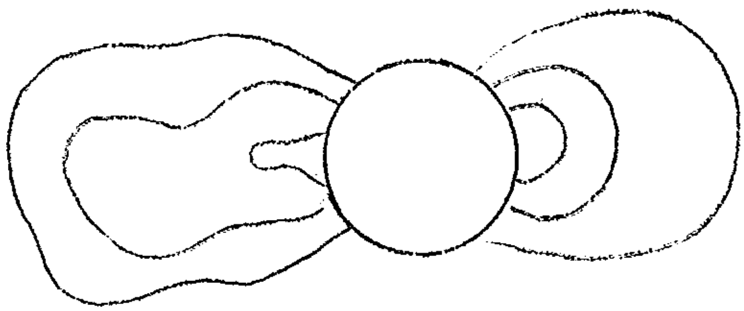

Fig. 13-2

Suppose you have a particle bouncing back and forth on a field line on the right side of the figure. Where will it be when it gets to the other side? Another invariant of the motion is needed -- the magnetic moment invariant only tells you that the mirror point is constant in a static field. The mirror point is an invariant of the motion, just like the magnetic moment is an invariant motion.

The other invariant of the motion that is needed is the integral invariant, and it is THE one of use to tell you where a particle will be as it drifts. It is an action integral. I' is defined as the integral between mirror points, $I' = \int_m^{m^*} w_{\parallel} dl$ (this definition will be modified later).

Now, since $w_{\parallel} = w \cos\alpha$, we can write

$$I = I'/w = \int_m^{m^*} \cos\alpha dl$$

or, since $\sin^2\alpha = B/B_m$,

$$I = \int_m^{m^*} (1-B/B_m)^{\frac{1}{2}} dl.$$

This is a form that only depends on the magnetic field. The particle motion is completely dropped out except we still must know the mirror points. We integrate along the field lines between mirror points, so the particle motion is still implicit.

If the integral I is a constant of the motion, then we can see there will only be one field line on the left side of Fig. 13-2 that can match a given field line on the right if the mirror points are fixed.

David Cummings

10/28/63

SPACE SCIENCE 500

Lecture XIV

Consider two particles moving in a perfect dipole field. Assume they start on the same field line but that they have different equatorial pitch angles so that they have different values of B_M and I. However, we can be sure if we start them out on the same field lines, since there is complete azimuthal symmetry for a dipole field, the two will stay on the same field line. And, once we define B_M and I for the single particle, we know not only where it will be every place on the other side, but we also know that another particle that starts out with a different I and a different B_M but on the same field line, will always remain on the same magnetic shell.

As a consequence, this is a degenerate system; once a particle is on a magnetic shell, it will stay on that shell all the way around.

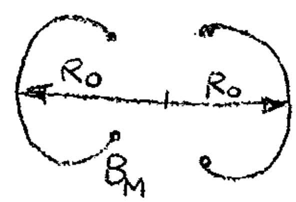

Fig. 14-1

This is not true for the earth's magnetic field. Look what happens if we distort the dipole just a bit. Consider particle #1 with its two mirror points so close together that the particle mirrors in the equatorial plane and drifts around in the equatorial plane. Thus, $I^{(1)}$ is zero. The other particle mirrors at $B_M^{(2)}$.

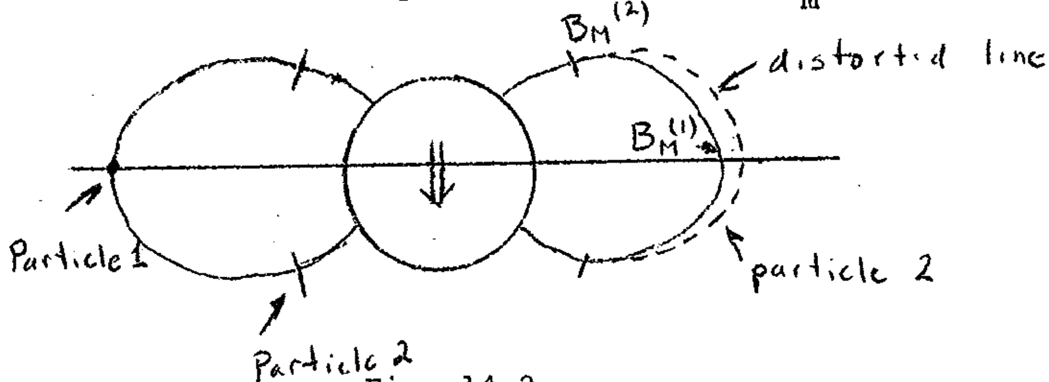

Fig. 14-2

Now let's assume that the perfect dipole field line is distorted a bit on just one side (as shown in Fig. 14-2) and all the adjacent field lines are distorted the same way so that the value of B at any point is not significantly different from that of the dipole, but the path lengths are somewhat longer. (We need to run a current through the field to do this.) Now, since I is $\int w_{\parallel} dl$ and since we have just stretched field lines a little bit, the particle, in order to maintain I constant, can no longer stay on the same field line. Particle (1), however, is unaffected by this distortion and thus remains on the same field line. When the field is deformed from a dipole, it is no longer a degenerate system. In other words, we have split the degeneracy; this is something like Zeeman splitting. If you have a degenerate system and apply some kind of external perturbation to it, you often split the degeneracy.

The other way we could write the integral invariant is

$$I = \int_{m}^{m^*} (1-B/B_M)^{\frac{1}{2}} \, dl \ .$$

Since we haven't changed the field strength and have just perturbed it a bit to make the field line longer, then, obviously, along the same field line the path length will be longer, but B(l) will be the same. Therefore, if we say that I is a constant of the motion, we must move closer to the dipole. If there is no azimuthal asymmetry, the degeneracy of the system is split.

But, as I mentioned before, the earth's magnetic field is fairly close to that of a perfect dipole. A common and useful magnetic coordinate system is the BL system suggested by McIlwain. For a perfect dipole, the distance R_o is the distance from the equatorial crossing of a field line to the magnetic center. We assume that there is an analogous distance L for the earth's field. If it is a degenerate system, L turns out to be R_o. The L value for a particle in an asymmetrical field is identical to the R_o it would have in a dipole field. Since the asymmetries in the earth's magnetic field are only of the order of 1%, L is always within 1% of R_o.

Thus, L is very nearly the distance from the equatorial crossing of a trapped particle to the magnetic center of the earth (see Lecture VIII, p. 2). If we give the L value and B_M for a given particle, its motion is completely described in the static geomagnetic field. McIlwain made the approximation that the earth's field is a degenerate system -- an assymption good to ~ 1% -- and thus relates a value of B (not necessarily B_M) and L for a particle at one point, and thus gets a useful, but not exact, magnetic coordinate system.

* * * * * * * * *

References for more information:

1. Northrup, T. G., and Teller, E., Stability of the Adiabatic Motion of Charged Particles in the Earth's Field, Phys. Rev., 117, 215 (1960).
2. McIlwain, C. E., Coordinates for Mapping the Distribution of Geomagnetically Trapped Particles, J. Geophys. Res., 66, 3681 (1961).
3. Stone, E. C., The Physical Significance and Application of L, B, and R to Geomagnetically Trapped Particles, J. Geophys. Res., 68, 4157 (1963).

David Cummings

11/1/63

SPACE SCIENCE 500

Lecture XV

The flux invariant is $\Phi = \int \bar{B} \cdot \bar{dS}$ inside an integral invariant shell. By integral invariant shell, I mean the shell traced out by a particle that drifts around the earth. If I take a surface -- an easy one to take would be the equatorial plane -- $\bar{B} \cdot \bar{dS}$ is evaluated through the equatorial plane, then the flux invariant states that this flux is a constant of the motion. You can see this intuitively: As a particle drifts around, it forms a ring that encloses some flux. The idea is similar to the one for frozen-in flux where the particles move with the field so that the flux enclosed is constant. We have a similar thing here. If I very, very slowly push the magnetic field lines together on one side, the integral invariant surface will distort so the particle drifts around to enclose the same total flux, Φ. (See Figure 15-1). The surface must be simply connected and the integral must include the flux that passes through the earth.

All three of the invariants of motion are derived in a rigorous manner in an article by Northrop and Teller: Physical Review, 117, 215 (1960).

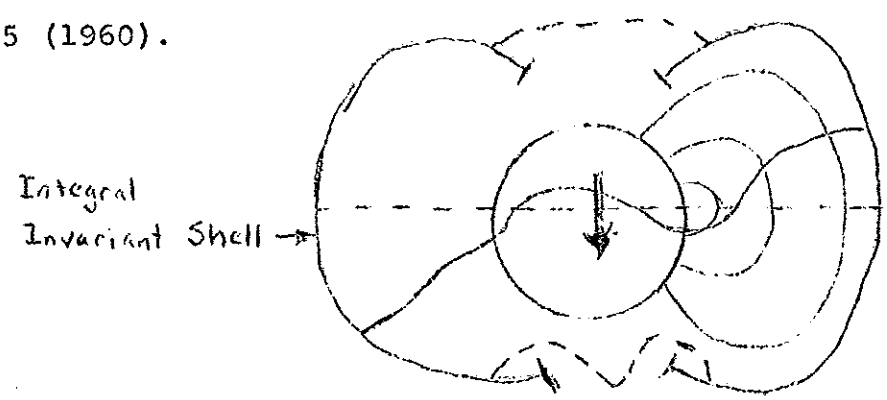

Fig. 15-1

The three invariants, then, are the magnetic moment invariant u, the integral invariant I, and the flux invariant Φ. What do each of them tell us? The first one u (equal to the perpendicular particle energy over B_o) requires that the mirror point be fixed in a time-stationary field. Thus, the mirror point is fixed. Let's say we have a dipole-like field on one side of the earth and very distorted field on the other side as shown in Fig. 15-2. Then let's draw a surface of constant B (dotted line),

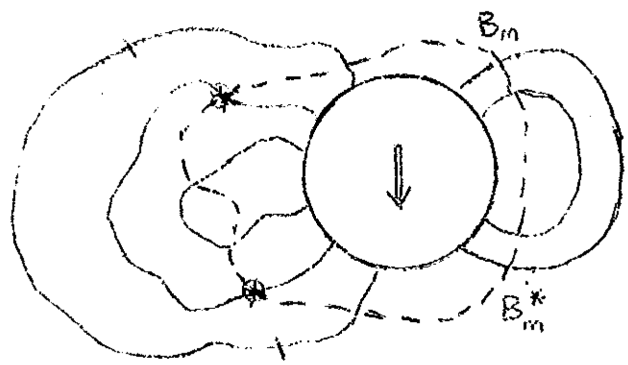

Fig. 15-2

and that is our B_M. The magnetic moment invariant tells us the mirror magnetic field is a constant. As the particle drifts around, it will always mirror on this surface of constant B.

The second invariant, the integral invariant ($I = \int w_{\parallel} dl$) tells us which one of these field lines is the right length, or which of these field lines we can expect the particle to appear on as it drifts around. (I is sometimes called the longitudinal invariant -- quite obvious nomenclature.) If you don't fix the B_M, then the integral invariant $\int w_{\parallel} dl$ could be satisfied on any field line if the mirror points are properly adjusted. However, if we fix the mirror points as B_M and B_M^*, then there is only one field line that truly satisfies the condition of constant I.

The third one is the flux invariant. The flux invariant ties the integral invariant shell to the magnetic field lines so if you move the magnetic field lines around, the integral invariant shell tends to move with the magnetic field.

To describe how the particles behave then in conditions that are distorted or conditions where the magnetic fields are changing with time, all three invariants are needed -- if they remain constant. But let's talk about where they might break down, because all of the invariants can be violated.

One way you can break down any of the invariants is to have a change in the magnetic field that is too fast for the particle to follow. If you have a magnetic field line with a particle spiralling around, and the field line moves in a time short compared to the cyclotron period, the particle won't be able to follow the field line motion; its magnetic moment won't remain constant. For example, if you have a uniform, constant magnetic field thru which a particle is moving, and then the field intensity suddenly goes thru a step-function change in intensity, the particle will run into this discontinuity and do something funny. The direction of motion will change very sharply, and the particle will end up with a different magnetic moment.

Remember from the general proof (Lecture XII), it was shown that the field lines must be straight in order for the magnetic moment to be strictly constant. Most books say that the field change should be small in one cyclotron. But that is not necessarily true. All that is required is that the field lines be straight. Any time they curve, slight perturbations of the magnetic moment result so that the magnetic moment oscillates about a mean value.

Let's talk about how easy it is to break each of them down. It is clear that it is rather easy to break down the flux invariant. A very high-energy proton close to the earth drifts around the earth in, say, 2 or 3 minutes; whereas a one kilovolt electron at 4 earth radii will take 5-6 days, a long time. If we make a change in the magnetic field, that is, very fast compared to six days (say, one day), it will break down the flux invariant for that particle. This one, for things like the Van Allen radiation, is practically useless. Many magnetic perturbations occur in periods of minutes or hours; a magnetic storm takes about a day. Thus, the flux invariant is relatively useless. The other two are quite helpful. The bounce periods for particles in the earth's magnetic field is of the order of seconds -- one second, ten seconds. That is rather shorter than most changes seen in the earth's magnetic field.

For the magnetic moment invariant the cyclotron period is the significant time scale. For an electron the cyclotron period is like microseconds, and for a proton it is a few milliseconds. So, in the increasing order of difficulty to break down the invariants by rapid changes in the magnetic field, the flux invariant is easiest, the integral invariant is second, and the magnetic moment invariant is the hardest to break down.

If the field line is curving, the particle can't really tell how the field is going to change. This gives us an interesting way to break down the magnetic moment with very small perturbations -- a resonant method of breaking down the magnetic moment. The particle moving thru the field isn't going to know how the field is going to change and it will first, say, under-correct; its magnetic moment will get too small, then it will recognize that it is converging too rapidly, the radius of the particle will spring out, and it will oscillate as it moves along.

Now let us wiggle the magnetic field at just the right frequency and put in a hydromagnetic wave so the field line is wrinkled with a wave-length lambda as shown in Fig. 15-3.

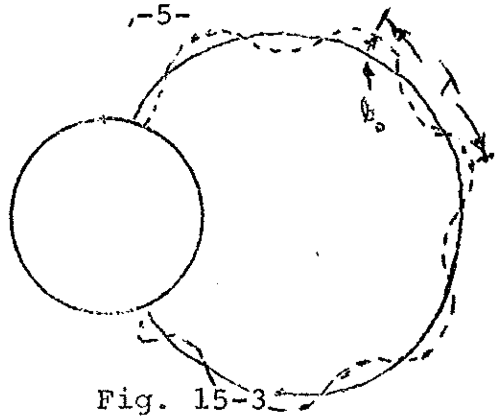

Fig. 15-3

If the particle moves along the field line so its motion is just 180° out of phase with these wiggles, it will be thrown back and forth, and energy will be transferred between parallel motion and perpendicular motion.

Another model that can be used for particles bouncing back and forth in the earth's magnetic field is the following: consider a trough with a marble rolling back and forth, and along the trough (Fig. 15-4).

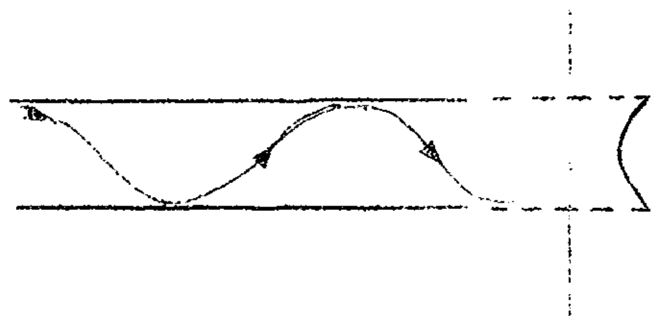

Fig. 15-4

It is clear that if I took this trough and put some wiggles in it so that every time the particle strikes the wall at a rather sharper angle than it would if the trough were straight. Thus, without any convergence of the trough at all, energy can transfer from parallel to perpendicular.

All that is needed to achieve resonant conditions is to get the wavelength for the hydromagnetic wave equal to the distance the particle will travel in a cyclotron period. So,

$$T = \frac{2\pi}{\omega_c} = \frac{\lambda}{w_{\parallel}} \quad .$$

Another relationship for the resonant conditions is as follows:

$$\lambda = \frac{V_{hm} 2\pi}{\omega_{hm}} \rightarrow \frac{V_{hm}}{w_{\parallel}} = \frac{\omega_{hm}}{\omega_c} .$$

$V_{hm} = \frac{\omega_{hm} \lambda}{2\pi} = \frac{\omega_{hm} w_{\parallel}}{\omega_c}$

The hydromagnetic wave, of course, is moving, but the wave velocity is 100 km/sec or so, and for example, Van Allen radiation particles move with nearly the speed of light.

Another way of looking at the resonant condition is that the moving particle doppler-shifts the wave so that its frequency moves up to equal the cyclotron frequency which is of the order of 10 - 100 cycles/sec for protons in the earth's field.

There are no 100 Mev protons observed in the outer zones of the radiation belt. This is a powerful mechanism that may well be responsible for removing protons (high-energy protons) that might be trapped in the outer zone of the radiation belt.

HOMEWORK

1. Assume a dipole field in which trapped particles are bouncing back and forth. What happens to the particles if the magnetic moment is increased very slowly? Just take the magnetic moment and crank it up very, very slowly so all the invariants are preserved.

2. Given the hydromagnetic wave velocity as in Figure 8-14 in the Satellite Environment Handbook, the ω_{hm} is 18 radians/sec, and the total velocity of a proton = 3 x 10^7 m/sec -- is there any range of L for which resonance is impossible regardless of the pitch angle alpha?

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11/4/63

SPACE SCIENCE 500

Lecture XVI

Vibrating mirror points are another mechanism which I will just mention in passing that can break down the integral invariant I.

If you have a dipole field with given mirror fields and fill the dipole with isotropic hydromagnetic wave radiation, the mirror point will move up and down as the wave comes thru. The mirror point is always B_M and that is equal to B of the earth's field plus b of the wave.

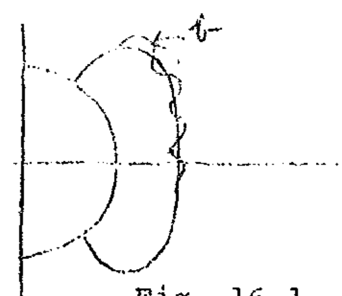

Fig. 16-1

When b adds to the earth's field, the mirror point moves up the field line; when b substracts from the earth's field, the mirror point moves down. So, when the waves go thru and a particle comes down to bounce against the mirror point, it doesn't bounce against a stationary wall or a stationary mirror point; it bounces against a moving wall or a moving mirror point. This is as if you threw a ball against a wall moving toward you -- it would bounce off with a velocity that was equal to its original velocity plus twice the velocity of the wall.

This is a second order effect here that results in an increase in the parallel velocity. Since h.m. waves have frequencies that are of the order of the particle bounce frequency, this could result in a cumulative effect. This actually changes the total energy of the particle; with an increase in the parallel velocity, the mirror point will walk down into the atmosphere.

A reference on this is E. N. Parker, JGR 66, 693 (61).

Now let's talk about the behavior of particle distributions in a non-uniform magnetic field. In particular, we can take as an example for simplicity a dipole field because that is one we will be most interested in.

Assume a distribution of particles in a non-uniform field such that at a certain pitch angle α and in a range $d\alpha$, we have a flux of particles $Nvd\Omega$ where N is the number density of particles, v is their velocity, and $d\Omega$ is the differential solid angle. The solid angle is

$$\Omega = \text{solid angle} = 2\pi(1 - \cos\alpha)$$

$$d\Omega = 2\pi\sin\alpha \, d\alpha \quad .$$

From u equals a constant, we know that

$$B_1/\sin^2\alpha_1 = B_2/\sin^2\alpha_2$$

where the subscripts refer to different points in the non-uniform field. Rewriting this,

$$B_1\sin^2\alpha_2 = B_2\sin^2\alpha_1 \quad .$$

The differential of this expression is

$$2B_1\sin\alpha_2\cos\alpha_2 d\alpha_2 = 2B_2\sin\alpha_1\cos\alpha_1 d\alpha_1 \quad ,$$

or, multiplying by π,

$$B_1\cos\alpha_2 d\Omega_2 = B_2\cos\alpha_1 d\Omega_1 \quad .$$

Then, $$d\Omega_2 = \frac{B_2}{B_1}\frac{\cos\alpha_1}{\cos\alpha_2}\,d\Omega_1 .$$

Now, if we have a tube of magnetic flux here, we know the flux of particles thru a plane at one point is equal to the flux of particles thru a plane at a different point if the trajectories are connected. Now, if I say that the number of particles in a pitch angle range $d\alpha_1$ is N_1 and A_1 is the area of the tube of flux, then the number of particles per unit volume per steradian is $N_1 A_1$ within $d\alpha_1$ at α_1. A similar result holds at point 2. The total flux thru a tube of flux is a constant for connecting trajectories, i.e., $A_1 N_1 d\Omega_1\, w\cos\alpha_1 = A_2 N_2 d\Omega_2 w\cos\alpha_2$. This equation expresses conservation of flux.

We also know that if u is constant, then the flux B times the area A is also a constant. We have shown that in a previous lecture. If we substitute $B_1 A_1 = B_2 A_2$ and the result for $d\Omega_2$ into the flux conservation equation, we get

$$A_1 N_1 d\Omega_1 w\cos\alpha_1 = N_2\left(\frac{A_1 B_1}{B_2}\right)\left(d\Omega_1 \frac{B_2}{B_1}\frac{\cos\alpha_1}{\cos\alpha_2}\right) w\cos\alpha_2 ,$$

which reduces to $N_1 = N_2$. Thus, for connecting trajectories, the particle distribution in phase space is invariant. The pitch angle changes as the area changes in such a way that they compensate beautifully.

I want to talk about the solar wind for a minute because Parker is coming, and if we don't talk about the solar wind before he comes, you won't get as much out of his talk.

It was recognized that there was probably a continuous solar wind of particulate or corpuscular radiation coming out from the sun when people tried to account for the outward flow of comet tails.

As comets orbit around the sun, the tail points away from the sun regardless of position. People tried to account for this by electromagnetic radiation (light) pressure, and found they couldn't account for it. The first guy I know that did this was Fitzgerald (of the Lorentz-Fitzgerald contraction theory), and he did it all wrong since he used radiation pressure against atoms non-quantum mechanically -- but his conclusion was right.

Then the next man to do it was Biermann, who also did it wrong. He neglected the interplanetary magnetic field and any plasma interactions, and used a charge exchange collision. First he showed electromagnetic radiation was not enough quantum mechanically; he did that right. Then he accounted for the tails by a charge exchange reaction where a particle, say, a proton moving along very fast comes near a neutral atom, and the electron jumps from the neutral atom to the proton with no exchange of momentum. In this you get a neutral hydrogen still moving fast plus a slow ion. This would ionize the cometary material. Once we get ions in there, e.g., a CO+ ion, then the coulomb collision cross-section is bigger and the ions can be swept back. However, subsequent experiments showed that the charge-exchange cross-section is orders of magnitude too small. It appears that the interplanetary magnetic field, plasma interactions, and things like that really sweep the tail back.

Parker looked at the problem and said, "Aha, we can treat this as a hydrodynamic expansion; that is, you have a hot gas and you just let is expand hydrodynamically, use the continuum approach and see what the equations of motion are for a gas at a temperature of a couple of million degrees (like the corona) and watch how it expands."

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DESSLER 11/6/63

SPACE SCIENCE 500
Lecture XVII

The comet tail problem convinced a lot of people that a continuous corpuscular radiation streaming out from the sun was needed. There are other things that would be difficult to explain without a solar wind. For instance, there is a cosmic ray diurnal variation -- that is, every day at a local time that depends on where the station is, the cosmic ray intensity will be a little bit greater (~ .3%) than at any other time. This is true no matter where the earth is in its orbit. It shows solar control, certainly not galactic control. It has nothing to do with the arm of the galaxy.

This solar control can be explained three ways: (1) The sun is a source of cosmic radiation, continuous, and no one really can justify that very well. It is sporadically known to be a source, solar flare cosmic rays, but not continuously in the energy range of up to 10^{10} or 10^{11} electron volts. Or, (2) the suggestion by Alfven that plasma comes out in beams with a vacuum on each side. These plasma beams modulate the cosmic rays. But, again, this is corpuscular radiation coming out from the sun. (3) Dr. O'Brien noticed a large diurnal variation in the counting rate of the Van Allen radiation at about 1000 km altitude with the Injun satellites. He pointed out this could be explained if you compress the field on the daytime side by the solar wind. Two explanations were offered. Either something is coming from the sun that can come into the field or a big belt of trapped particles is deforming with the field so they dip down a little lower on the daytime side and are pulled up a little higher on the nighttime side so that at 1000 km there is a big diurnal variation. The exact mechanism is not important.

The reasoning, which I think is very sound, is there is something from the sun that is causing the diurnal variation in trapped particle flux.

Now let's talk about Parker's approach to the solar wind. First of all, he was convinced that there is a solar wind, that the corona is continuously expanding. He noted the thermal structure of the corona as shown in Figure 17-1. The temperature starts

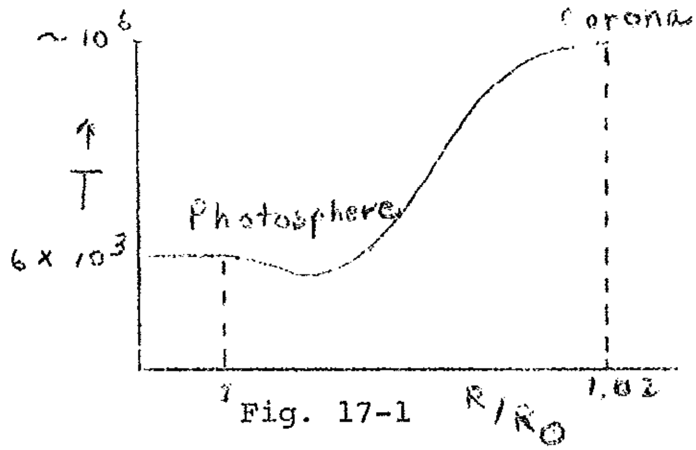

Fig. 17-1

out at 6,000°K at the photosphere and, at first, as you go out, the temperature is either constant or perhaps falls a few hundred degrees. Then it begins to rise and very quickly rises to 1 to 2×10^6 degrees. One thing is clear right away. Heat is not conducted up from the photosphere into the corona; if anything, heat is conducted down from the corona. This means that the corona has its own heat source; something is putting heat into the corona to keep it hot. This something probably originates in the sun. It is quite unlikely that it would come from some place outside and concentrate around a star. It comes from within the sun and it can pass thru the cool region just above the photosphere without dissipating any heat. (Reference: Astrophysical Quantities by Allen gives a table of temperature versus radial distance.)

The heat source is thought to be acoustic waves that, as they move out, turn into hydromagnetic waves -- they just transform as the particle pressure diminishes, the magnetic field pressure takes over. Farther out above the photosphere they would tend to form shock waves because the total pressure is falling and the amplitude of the wave will be roughly constant as it propagates out. When the wave pressure over the total pressure is greater than one, you expect to find a shock wave. Then you would expect heating from these shock waves.

Now, the difficulty is that this is a collision-free shock, and no one knows how to get a dissipation mechanism when there are no collisions. But the idea is that close to the sun $\Delta P/P$ is less than one; in the corona $\Delta P/P$ is greater than 1. Where they form shock waves, the unknown dissipation mechanism takes over and dumps the energy of the wave into the corona. Since the collision mean-free-path is rather long in the corona (it's a tenth of an AU or so), heat will be conducted out very readily. Also, the waves that grow to form shock waves will develop at different distances from the sun. Thus, we can expect the heat source to be extended out thru the corona. It seems unlikely that there is a thin layer where all the energy to heat the corona is deposited. Thus, we can expect coronal heating to be extensive; we will see later why this is important.

It was once thought by some that solar prominences, in which a little piece of the sun goes flying off into space, supplied the necessary solar corpuscular radiation. However, the escape velocity from the surface of the sun is over 600 km per sec, and almost never does prominance material move out with a velocity greater than the escape velocity. Something more continuous is needed that doesn't involve explosions throwing things out.

It is clear that these catastrophic explosions are not the sort of thing that are going to explain comet tails, solar wind observations and things like that; they are so infrequent that they just can't account for the observations. Parker looked at the coronal expansion as a continuous phenomena. We have a ball of hot gas at 1 to 2 x 10^6 °K in a solar gravitational field. Let's look at what happens by applying hydrodynamic expansion argument.

Before we do that, I am going to talk about a rocket nozzle and how you get supersonic flow in a rocket engine. To get supersonic flow from hot gas, it must expand thru a deLaval nozzle. Parker has an analogy where he uses solar gravity in the corona to take the place of the rocket nozzle so that the solar wind expands supersonically.

Let's start out, then, with some hot gas in a container and put a little pipe in the side of the container so the gas can come out. As the pressure in the container is increased, the velocity of flow will increase until it reaches the speed of sound. That is, the flow velocity will be limited to the speed of sound. The speed of sound a for a monotonic gas is

$$a = \frac{v_{rms}}{\sqrt{3}} \sqrt{\gamma}$$

because the particles spend a third of their time going in each of three directions. The square root of gamma takes into account the fact that V_{rms} increases a little bit at the peak of the wave because when the gas is compressed in a sound wave, it heats up a little bit. In fact, the pressure, on the average, is higher every place because the compressions do not quite cancel the rarefactions (remember that radiation pressure is just the average pressure).

Can the molecules get out of the hole any faster than their thermal velocity? DeLaval wanted to get a steady flow of gas at the highest possible velocity. To do this, he invented the deLaval nozzle.

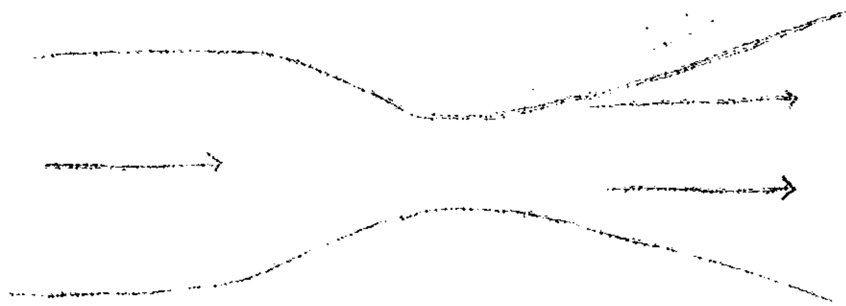

Fig. 17-2

Now I am going to show you a mathematical proof that the deLaval nozzle is necessary to obtain supersonic flow. I am sure this is what was done AFTER deLaval thought of the idea, not before. You will see that you would never be able to get at it this way. The answer is, interestingly enough, to get the highest possible velocity, you need a converging-diverging section. This is why rocket motors are made as shown in Figure 17-2. At the throat, the velocity is limited to the velocity of sound. Then as it moves out, it will become supersonic. The total energy of the system has to be constant. So, the transverse energy appears as directed energy. All the deLaval nozzle does is take the random thermal motion and direct it. The exhaust velocity is equal to V thermal on the upstream side of the combustion chamber. If the velocity does not reach the speed of sound in the throat, then the nozzle is just a Venturi tube. The gas slows down again as it goes out. There was a big argument between Parker and Chamberlain that went on in and out of the literature for a couple of years because there are two solutions as sketched in Figure 17-3.

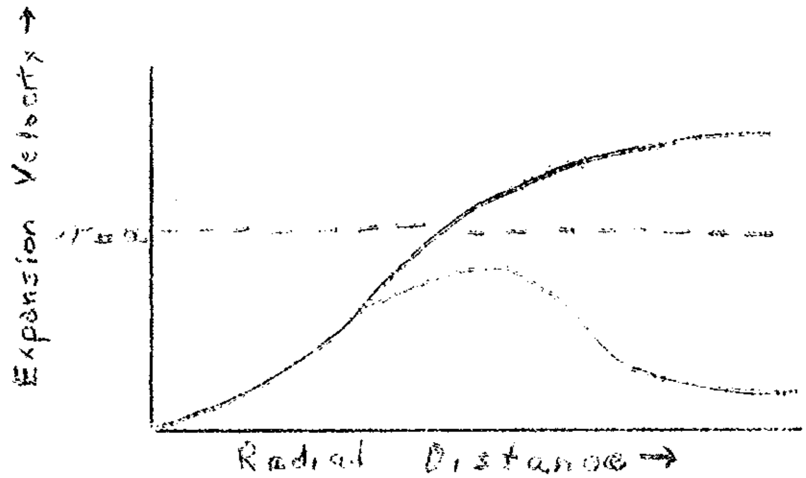

Fig. 17-3

One works as a Venturi tube and the other works as a supersonic nozzle. Chamberlain got a velocity of expansion of only 20 km/sec and Parker got an expansion velocity of 500 km/sec -- they weren't agreeing at all.

Let the cross-sectional area be f at any point along the nozzle. The Mass flow M is $f\rho V$ where V is the flow velocity, f is the cross-sectional area, and ρ is the density at any point. This is the continuity equation, $M = f\rho V$. Take the logarithm of both sides and differentiate. M is a constant for steady state. Thus,

$$\frac{df}{f} + \frac{d\rho}{\rho} + \frac{dV}{V} = 0 \quad .$$

Now from Bernoulli's equation,

$$dP = - \rho V dV \quad .$$

Or,

$$\frac{dP}{\rho} = - V dV \quad .$$

(The change of pressure is balanced by the change in momentum.)

$$\frac{dP}{d\rho}\frac{d\rho}{\rho} = - VdV \quad ,$$

or, since $dP/d\rho = a^2$,

$$a^2 d\rho/\rho = -VdV \quad .$$

Therefore,

$$d\rho/\rho = - \frac{V}{a^2}\, dV \quad .$$

$$\frac{df}{f} - \frac{V}{a^2}\, dV + \frac{dV}{V} = 0$$

Substitution of this result in the differential form of the continuity equation gives

$$\frac{df}{f} = \frac{dV}{V}\left[\frac{V^2}{a^2} - 1\right]$$

$$\frac{df}{f} = \left(\frac{V^2}{a^2} - 1\right)\frac{dV}{V} \quad .$$

In terms of the Mach number $M = V/a$, this becomes

$$\frac{df}{f} = (M^2 - 1)\frac{dV}{V} \quad .$$

Now, if the velocity is to always increase -- that is, if the velocity is always positive as we would like -- this is always positive. Then when the Mach number is less than one, df/f must be negative, that is, the throat must be convergent if the Mach number is less than one. When the Mach number equals one, df/f must be zero. When the Mach number is greater than one, df/f must be positive and the rocket nozzle is divergent. This shows that if we want to get a Mach number greater than one, we need to have first a converging section, and then a section with no change in area, and then a diverging section.

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DESSLER 11/8/63

SPACE SCIENCE 500

Lecture XVIII

The temperature of the corona is of the order of 10^6 degrees. This corresponds to roughly 100 ev. At the orbit of earth the solar wind protons have a velocity of 500 km/sec, which corresponds to an energy of 1.3 kev, which corresponds to the temperature of -- well, 1 ev is 12,000 degrees, so it is about 2 or 3 x 10^{7o}K, and they are just moving a few km a second. When they get out to the orbit of earth, they are moving 500 km/sec, which corresponds to 3 x 10^7oK.

Remember that in a simple rocket engine, the best you can do is to take the thermal velocity and order it. In the solar corona, we quite obviously have exceeded the thermal velocity by maybe a factor of 4. The solar wind energy comes from heat put in at the base of the corona, and this heat is converted into ordered motion. In the solar wind, the gas is heated after it has started moving. The Parker model just uses heat that is conducted out from the base of the corona. There also could be heat dissipated in a broad region many solar radii thick -- energy brought up from below in wave motion. The energy per atom at infinity, or very far from the sun, 1/2 mv^2, should be equal to the quantity of heat energy put in, minus the work done against gravity. There is a certain amount of thermal energy put in, and this should all appear as ordered kinetic energy and potential energy.

There is a lot of work left to be done on Parker's model. It doesn't make a good fit to the observed data; there are lots of adjustable constants, and there are things that need to be examined more carefully and, perhaps, modified in cases.

Let's look at the solar magnetic field. (Parker and Chamberlain excluded the solar magnetic field from their discussions.) If the solar magnetic field energy density is greater than nkT, there will obviously be no expansion. The magnetic field lines which start in one region of the sun and go to another, will trap the plasma. The heat put into the corona will be conducted back down to the photosphere where it is 10^3 times cooler, and it can be radiated away; there will be no energy carried away by an expanding corona. If the magnetic field energy density were greater than the particle energy density, you get no expansion. The heat that was put into the corona would then be lost either by conducting heat downward to the cooler photosphere or by radiating it away in electromagnetic radiation -- or both.

We must now solve for the magnetic field around a rotating sun. Let's first assume that the field is very weak, and we will assume initially that $B^2/2\mu_o$ is much less than nkT. We are looking down on top of the spin axis of the sun and the sun is turning.

Fig. 18-1

We will watch a block of matter move radially out from the sun. These are successive positions of this piece of matter. Meanwhile, the sun turns so that these blocks of matter move out radially. Each one carries a magnetic field frozen in with it. Let's follow the history of one of these pieces of matter as shown in Figure 18-1. A block of solar corona material that moves radially out will trace an Archimedes spiral as seen from the sun. Thus, as seen from a stationary frame, a single spot on the sun shoots out plasma that at any instant of time forms an Archimedes spiral. The frozen-in solar magnetic field also forms an Archimedes spiral. The equation for an Archimedes spiral is $R = (V_S/\Omega)\theta$. V_S is the solar wind speed,

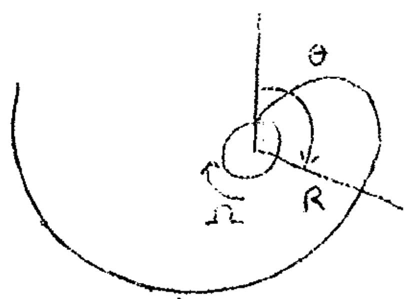


Fig. 18-2

Ω is the angular speed, and θ is the heliocentric longitude of the sun. Since $\nabla \cdot B = 0$, there must be tubes of flux going out from the sun, and tubes of flux coming back to the sun. So, if you take a surface integral over the sun of B, it is zero.

I will bring up the phonograph needle analogy because it might be helpful. If you have a phonograph record with a spiral groove (the phonograph groove is an Archimedes spiral, a rather tight one), the needle slides along the groove just like the plasma can slide along the magnetic field but not across it. The needle moves radially out (actually, it moves radially in) while the grooves rotate as a rigid body. The magnetic field lines are in the form of an Archimedes spiral that on the average co-rotate with the sun while the plasma slides radially out.

For a homework problem prove that, in the equatorial plane of the sun, the total field B_t is

$$B_t = B_o\left(\frac{a}{R}\right)^2\left[1 + \frac{\Omega^2 R^2}{V_S^2}\right] ,$$

where a is the radius of the sun and B_o is the field strength at R = a. The radial component $B_\parallel$ and transverse component $B_\perp$ are

$$B_\parallel = B_o \left(\frac{a}{R}\right)^2 ,$$

$$B_\perp = B_o \frac{a^2 \omega}{RV_s} .$$

Note that Ω (that is, the angular velocity of the sun) is not the synodic period, the apparent period. It is a period that corresponds to the 24.7 days actual rotational period that most people plugging into this use rather than the usual 27 day period, the apparent synodic rate as seen from the earth. Obviously, the motion of the earth around the sun has nothing to do with the strength of the interplanetary magnetic field.

Now we must go back and see if we have violated anything of physical significance that might prohibit the solar wind from expanding. We can define the ratio β as

$$\beta = \frac{\frac{1}{2}\rho_S V_S^2 + nkT}{B_t^2/2\mu_o} .$$

Far from the sun, the first term of the numerator will be dominant. Close to the sun, the second term will be dominant. If beta is less than one, there will be no solar wind. It turns out that it just barely works.

I will just give you a couple of numbers: between 1.1 and 1.2 solar radii there are 10^8 particles/cm^3, the temperature is a couple of million degrees, and nkT is 3×10^{-2} ergs/cm^3 while $\frac{1}{2}\rho_S V_S^2$ is only a small addition to this energy density. The magnetic field energy, $B^2/2\mu_o$, is 4 x 10^{-2} ergs/cm^3. Thus, B appears very slightly greater than 1 -- we can just barely get away. If the fields were, say, 3 gauss at the surface of the sun, the steady solar wind would not start anywhere near the base of the corona. You can see that just from a simple examination like that.

HOMEWORK PROBLEM

β is a function that depends on solar wind velocity. What is the minimum velocity a solar wind must have in order that it can expand continuously and not be held back by the solar magnetic field?

If you go far from the sun and take a high solar wind velocity, B_t will vary as $1/R^2$. Since the mass density decreases like $1/R^2$,

$$\beta \sim \frac{1/R^2}{1/R^4} = R^2 \quad .$$

So, for very high velocities, once β is greater than one anyplace, it will be greater than one everyplace. But, as we come down to lower and lower velocities where the $B_\perp$ component becomes the major one, then, since $B_\perp = B_o a^2 \omega / RV_S$, it is seen that

$$\beta \sim V_S{}^4/B_o{}^2 \quad .$$

David Cummings

DESSLER 11/11/63

SPACE SCIENCE 500

Lecture XIX

This morning we are going to take up what finally happens to the solar wind and, also, the solar magnetic field. In other words, what are the ultimate fates of the solar wind and the solar magnetic field?

I have here a 1961 paper by Parker, Astrophysical Journal, July, 1961, 134, page 20. Let me read a small part of what he says: "A number of questions have been raised recently concerning the properties of a stellar wind, in particular, the solar wind, at large distances from the parent star." And, there is a footnote: Fourth Symposium on Cosmical Gas Dynamics, Varena, Italy, August, 1960. So, at the end of 1960, people were asking what happens to the solar wind at large distances from the parent star; what is the ultimate fate of it? In this paper Parker concluded that if you let the solar wind go out far enough, it will become so diffuse that it will just be swept away by the galactic wind. However, we will adopt a different approach. First we will assume that the galactic magnetic field is of primary importance in determining the position of the interplanetary shock transition where the solar wind runs into an obstacle -- the galactic magnetic field. Setting $\frac{1}{2}\rho V_S^2$ equal to $B_g^2/2\mu_o$ where B_g is the galactic field strength, we obtain

$$\frac{\rho_E V_S^2}{S^2} \simeq \frac{B_g^2}{2\mu_o} \quad \text{or} \quad S \simeq \frac{V_S}{B_g}\,(2\mu_o\rho_E)^{\frac{1}{2}} \quad ,$$

where ρ_E is the solar wind mass density at the orbit of earth, and S is the heliocentric distance in A.U. S is where the solar wind undergoes a transition from supersonic to subsonic flow. The solar wind can push aside the galactic magnetic field. Locally, in the region around the sun, the galactic magnetic field will be bowed out. Leverett Davis showed in 1955 that this was the sort of picture that the solar wind would make in the local galactic magnet, make a bubble in it probably more like a football shape.

Remember what I said about the surface of the sun, that there were rather large regions of north magnetic field and of south magnetic field, so we have field lines going out and, also, field lines coming in. Now, when you go thru the shock, this spiral really tightens up because the flow out becomes subsonic behind the shock, and the field lines are oppositely directed in tubes of flux throughout this region.

David Cummings

DESSLER 11/13/63

SPACE SCIENCE 500

Lecture XX

SWEET'S MECHANISM

Last time we set up the problem that at large distances from the sun, after we have a shock transition for the solar wind, we have a condition where opposed field lines lie adjacent to each other.

I might say that no one knows what really happens, and there are other possibilities, some of which haven't even been considered yet, I am sure. But one attractive possibility is to merge these field lines so the field is annihilated. To do that, we need to have a sufficiently low conductivity -- the length of time required to merge them will be a function of the conductivity. If we have a low conductivity, it will go quickly, and if the conductivity is very high, it will take a long time.

There is something different about this system; you've got a region where the magnetic field is in one direction and another region where the magnetic field is in the opposite direction. In between, there must be a thin sheath where the magnetic field is zero. And this is where we come up to "Sweet's mechanism".

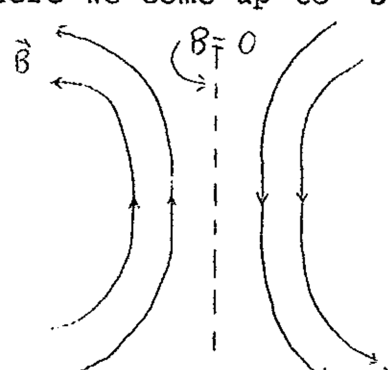


Fig. 20-1

When we worked out the previous magnetic diffusion problem for a magnetic field to diffuse a distance l, we found the characteristic time was approximately (and this would depend upon the geometry) $\mu\sigma\ell^2$ where σ is generally σ_3.

Now, let's set up some dimensions as shown in Fig. 20-2. We will assume a finite conductivity, so there is a merging velocity U. The field is annihilated as they move together. We can write the merging velocity $U = \ell/t_m$.

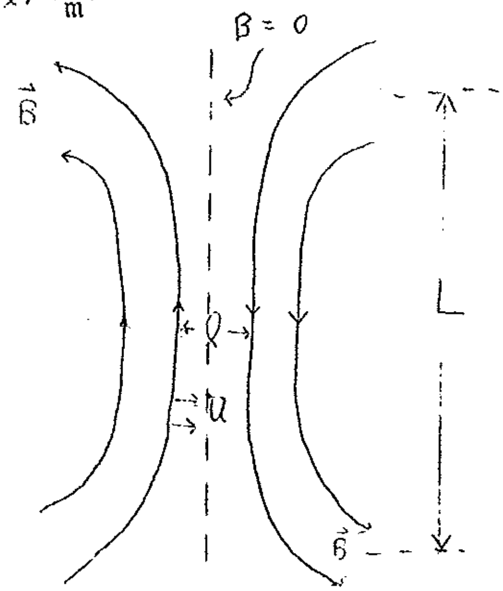


Fig. 20-2

$$\tau = \mu\sigma\ell^2$$

$$U = \frac{\ell}{t_m} = \frac{\ell}{\mu\sigma\ell^2}$$

$$U = \frac{1}{\mu\sigma\ell}$$

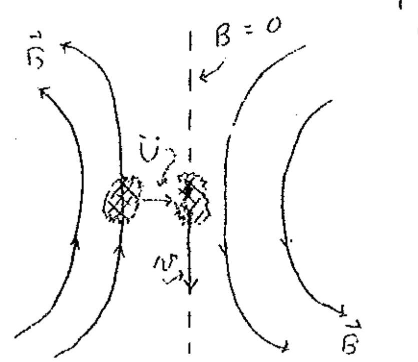
The σ should be σ_3 because in this case there is no Hall current. You can quickly prove σ should be σ_3.

If we have these magnetic fields moving together, we are going to have matter piling up in the middle that will have to squirt out the ends. We have to get rid of the matter that piles up. So, we will let v equal the velocity of efflux of the matter that squirts out. And just by geometrical arguments v is going to equal U times L/ℓ. You can see this; if the particles have to move out a distance L, they are going to have to go a higher velocity than the merging velocity by just this ratio.

$$v = U\frac{L}{\ell} = \frac{L}{\mu\sigma l^2} \quad .$$

We have here assumed hydrostatic equilibrium and nkT much smaller than $B^2/2\mu_o$.

Let's follow the life history of a little block of plasma in the zero field region. As it moves into the middle where there is no magnetic field and then gets squirted out, it starts out with zero velocity and then is accelerated to a velocity v. This means that there is a reaction pressure, $\frac{1}{2}\rho v^2$, which must be balanced by the magnetic pressure $B^2/2\mu$ -- in other words, the magnetic field squirts the particles out.



Plasma squirted out by magnetic pressure.

Fig. 20-3

$$\frac{1}{2}\rho v^2 = \frac{B^2}{2\mu_o}$$

$$v = \frac{B}{\sqrt{\mu_o\rho}} = V_{hm} \quad .$$

But notice, interestingly enough, that the efflux velocity is just the hydromagnetic wave velocity. We could have written this down immediately without going thru this step because you just can't move material thru a magnetic field containing a plasma any faster than the hydromagnetic wave velocity. You might for a time have a shock wave implying a somewhat greater velocity, but that would require a large energy source. Thus, $v = V_{hm}$. Then we write down that the merging time by Sweet's mechanism is L/U. We really want to merge a field of scale, L; the little distance here of ℓ is not significant, so that L/U is the merging time. We don't really know what this little "ℓ" is. It was just an arbitrary scale length where we said the field was so weak that the particles would just come squirting out.

Then,

$$t_m = \frac{L}{U} = \mu\sigma\ell L = \frac{\mu\sigma\ell^2 L^2}{\ell L} .$$

$$\ell = \frac{1}{\mu\sigma V}$$

$$\text{But } v = V_{hm} = \frac{L}{\mu\sigma\ell^2}$$

$$\therefore \quad t_m = \frac{L^2}{V_{hm}\ell} = \frac{L^2}{V_{hm}} \left(\frac{\mu\sigma V_{hm}}{L}\right)^{\frac{1}{2}} = \frac{L^2}{V_{hm}^{\frac{1}{2}}} \frac{(\mu\sigma)^{\frac{1}{2}}}{L^{\frac{1}{2}}} \frac{L^{\frac{1}{2}}}{L^{\frac{1}{2}}}$$

$$t_m = \left(\frac{L}{V_{hm}}\right)^{\frac{1}{2}} (\mu\sigma L^2)^{\frac{1}{2}}$$

This is the merging time by Sweet's mechanism. It is very interesting in the form that we have gotten it. L/V_{hm} is the time for a hydromagnetic wave to travel a distance L. And the second term is the time for what we call the old-fashioned diffusion . Thus, t_m is the geometrical mean of the two times for the merging over scale L by ordinary magnetic diffusion and the hydromagnetic wave propagation time. This process is much faster, as you might expect, because you annihilate the field and squirt out the matter.

One final and very important point: This process cannot go any faster than the collision period. That is, if the merging goes at a rapid rate, then the conductivity will have a large imaginary component. Then energy will be stored, and the merging will stop. Whenever you finish a magnetic diffusion or Sweet's mechanism calculation, always check to be sure that $t_m >> 1/\upsilon_{collision}$.

David Cummings

DESSLER 12/6/63

SPACE SCIENCE 500

Lecture XXI

This morning we are going to discuss the effect of the solar wind striking the earth's magnetic field. This is sometimes called the Chapman-Ferraro problem.

Consider a perfect undistorted dipole field and bring up a superconducting plate as shown in Figure 21-1.

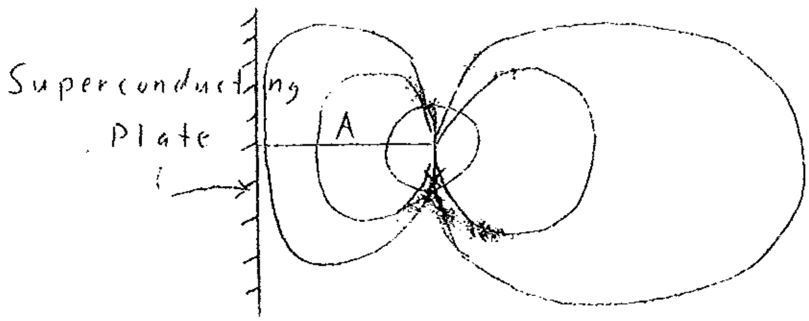


Fig. 21-1

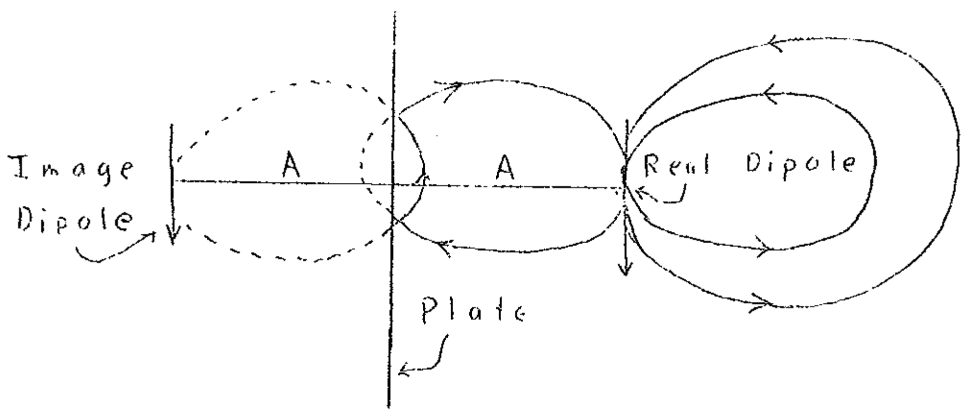


Fig. 21-2

I am going to go thru this once using superconductors, and then we will switch over to solar plasma, very cautiously, because it doesn't necessarily transform. You can't just go from one to the other without a little care.

I bring this plate up, and we know that inside a superconductor there can be no magnetic field -- it completely excludes any impressed magnetic field. And that means, since $\nabla \cdot B = 0$, B normal is zero every place on the plate. There is a simple solution for this magnetic field configuration. If the dipole is a distance A from the plate as shown in Figure 21-2, we just put an image dipole a distance A inside the plate. Since these dipoles are exactly the same strength, one is the image of the other, they are the same distance from the interface, and the normal components will be cancelled out. The normal components all cancel out, and only the parallel component is left. But, a parallel component is alright. I can take H·dl around the surface of the plate, and that just means I have a surface current, <u>no</u> magnetic field inside.

I want to say this about the image: It is just a virtual image; there is really no magnetic field inside. We have the field of an image outside the superconductor, but the image doesn't produce a field inside. So, H·dℓ around the surface gives the surface current that must flow in the superconducting plate to cancel the field inside the plate.

Now let's redraw the dipole field as squashed in by the superconducting plate. This is shown in Figure 21-3.

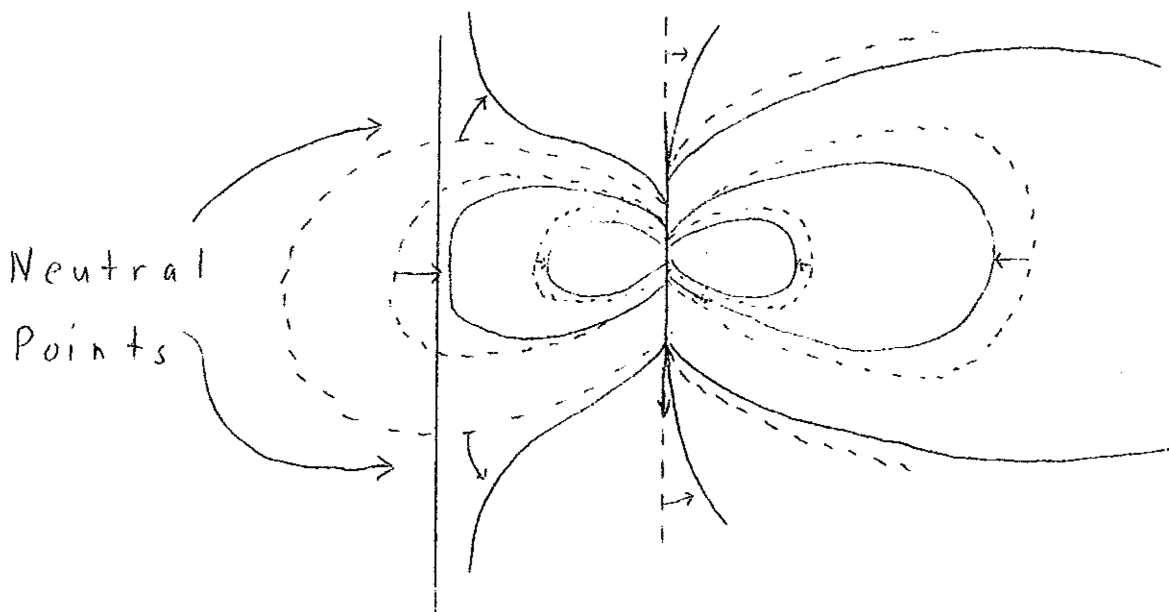


Fig. 21-3

When you compress the field on one side, you compress it everyplace. I can just poke a superconducting finger in, and all the field lines everyplace will move closer together because it has some of the properties of a hydrostatic medium. In the case of the real earth's magnetic field, when I compress this, the only way the field knows that I have compressed it is the hydromagnetic wave propagates thru the magnetosphere and carries the word to the tail-end that there has been a compression in the front. The field lines move closer together, the word travels on, and they all scrunch in. So, when you compress the front side, the field lines on the back side move in, although not as much. You are farther away from the image dipole. This effect is shown in Figure 21-3 where the dotted lines show the unperturbed dipole field while the solid line shows how they are deformed. The arrows signify motion of the field lines. The field line that would have gone to infinity now goes out some distance, closes, and comes back. It acts as a stretched rubber band, and it pulls in on the back. This is where the compression on the back side comes from. All the field lines that would have gone out on the front side are now coming around to the back, all pulling in. Remember, the curvature of a field line has to be balanced by a pressure gradient, so all these field lines here pull in and compress the field on the back side.

The point then, to summarize this, is when the plate is pushed into position, the field strength in the equatorial plane is increased everywhere. This means the contours of constant B must move out because you are increasing the field strength everyplace.

Now let's look at the shape of the contour of constant B. First of all, even looking at it from our image dipole or from a current flowing on the plate, it becomes immediately obvious that the constant intensity contours are not symmetrical around the earth anymore. The field strength next to the plate, on a line connecting the two dipoles, is doubled because the field of the dipole and the field of its image are equal, and they add. Everyplace else in the equatorial plane the field is increased, but less than doubled. That means the contours of constant B, to exaggerate, go as shown in Figure 21-4.

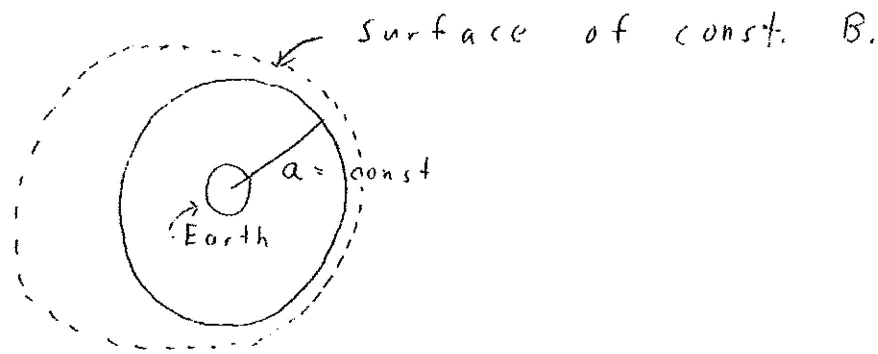


Fig. 21-4

If we have particles drifting around with a constant magnetic moment, let's say they have a 90-degree pitch angle, they move along a surface of constant B because their total energy is a constant and u is a constant. The radiation will drift closer on the back side than the front side. This is the case where we have a plane superconductor. Now, for the case of the solar wind striking the earth's magnetic field, for one thing, it isn't going to be plane because the solar wind pressure, which is ρv^2, is going to be constant all across the plane surface. But $B^2/2\mu$ is obviously going to be less the farther we move away from the center line. The solar wind pressure can push in farther at all other points. And that means, instead of using a plane, we know that the surface is going to be deformed like this.

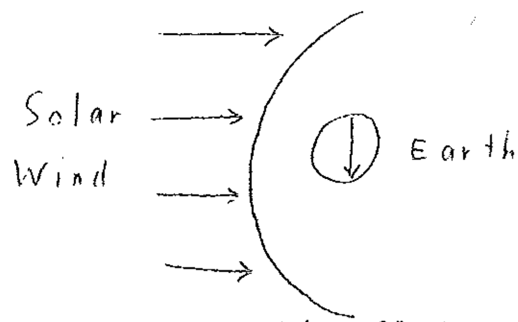


Fig. 21-5

There are several papers calculating the shape of this surface where they take a dipole and the solar wind, ρv^2. They assume specular reflection (and we will talk about that assumption later; it is not too good an assumption). They get a normal pressure on the surface, which depends, of course, on the angle the surface makes with the incoming solar wind. Then you calculate how much the field is increased inside and balance ρv^2 times the cosine of theta against $B^2/2\mu$. Actually, you put a $2\rho v^2$ out here because of the specular reflection. This is a difficult problem, but several people have calculated, to three or four significant figures, the shape of the surface. Probably the best one is the one by Midgely and Davis that appeared in JGR 68, 5111 (1963). That is the best analysis, I think. But the assumptions that go into it are crude enough that just the general picture is correct. I am sure that the surface could be off an earth radius or two, and they would never find it by that method.

It is pretty clear that instead of a plane, as the solar wind strikes the magnetic field, we expect the surface to bend around and give a comet-like shape to the magnetosphere. But the effect of the compression is the same.

There is something again to point out that is interesting: The last field line that closes on the front side is very close to the auroral zone, but I don't see what that has to do with the aurora. The aurorae occur most intensely at night, and those field lines are deeply buried within the earth's field. The fundamental understanding of the mechanism of the aurorae is probably the outstanding question in space science today. There is something really big that is going on out there that isn't understood in principle, but it occurs on these field lines here that are very heavily shielded by more field lines outside. There are a lot of auroral theories where they take so-called neutral points (see Figure 21-3) where the field changes direction, and they squirt in particles and they could make a big aurora at noon but not at midnight when it is observed. The sky is dark one or two hours after sunset; the sky is about as dark as it ever gets and you just put a photomultiplier looking up at the night sky. Or at a place like northern Denmark or Sweden during the winter, it is dark for six months, and you see the aurora at any time. Within an hour of midnight the intensity appears to peak. If you put in particles on the front side and then drift them around, how would you get them out at midnight? Why not out at some other time? That doesn't help at all. It is no easier to inject particles at neutral points than anyplace else. Magnetic pressure is isotropic. It is just as hard to push the field lines apart one way as another. I just wanted to bring up the neutral point because it is an interesting feature. It might be a coincidence or it might have something to do with the aurora, but without a theory, we will never know.

The picture I have drawn here is quite all right for a superconducting plate. Now, if we are going to use a plasma striking the field, then we have to worry about edge effects. To illustrate the principle, let me give a simple example shown in Figure 21-6.

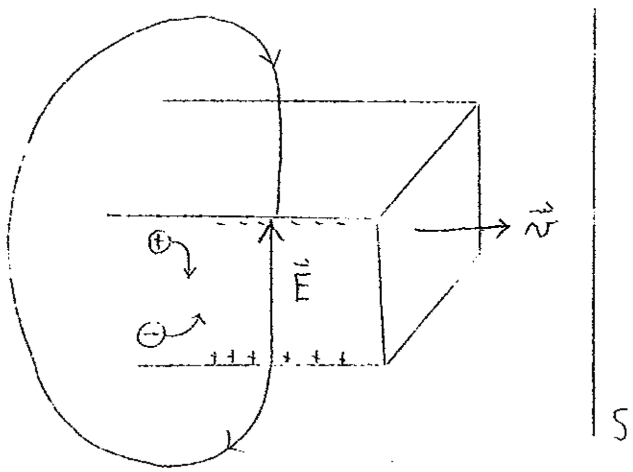


Fig. 21-6

Say I have a B field coming out of the paper on the right side of the surface S (i.e., B is restricted to the right side of the semi-infinite plane, S). A square cross-section column of plasma strikes the plane. Now, as it does this, the plus charges will move slightly to one side and the minus charges slightly to the other side. You will get a charge separation and electric field inside the plasma. As seen from inside the plasma -- this is a collisionless plasma so that $\vec{E} + \vec{v} \times \vec{B}$ will be zero -- this charge separation will continue until there is no more electric field inside. In other words, the electric field seen by a stationary observer will build up until it is equal to the $\vec{v} \times \vec{B}$ electric field. In that case all the forces balance, and this plasma column goes right thru just as if the magnetic field were not there. Inside the plasma you could see there is a magnetic field, you could measure it, but you would think you were standing still relative to it. I mean, if you had a magnetometer, you could tell that you suddenly entered the region of the magnetic field but if you were inside the plasma and moving with it, you would detect no electric field. And, therefore, you would say you were standing still, and it would be because of the surface charge.

Now, if we put plasma to the right of S, it will interact with the column. Because not only is there an electric field inside the plasma, but there is also an electric field outside. This electric field outside will set up an E cross B drift for all the plasma it goes thru. What will happen is that all the plasma starts moving, the column will stop, and a hydromagnetic wave will propagate out. (See Alfven, p. 77).

You can also think of a metallic satellite moving around the earth. Inside the satellite there is no electric field due to V cross B motion because the satellite is metal and you know that inside a metallic conductor there is no electric field. As the satellite moves along, you get minus charges on one side and plus on the other, and these just cancel the V cross B electric field that is set up by the motion thru the earth's field.

Finally, suppose I had a laboratory magnet, 12" pole piece, and I drop a glass sphere between the pole pieces. When it goes thru, there is an electric field across the glass; no charges flow because the glass is an insulator, but the electric field is there.

Don't expect me to develop in the class every possible way of looking at everything, and every possible answer because there isn't time, and I am not that patient -- and it won't do you a lot of good. The main thing is that, in order that you understand these things, you have to ask yourself questions and develop the ideas yourself.

David Cummings

DESSLER 12/9/63

SPACE SCIENCE 500

Lecture XXII

We will now investigate the effect of charge separation on the Chapman-Ferraro problem. Instead of the expected curved surface, we will have the earth's field inside a box or parallelpiped. It will have square edges and this will make it a little easier to see what the effect is. Then we can go back and look at the real case.

The plus charges come thru and are bent out of the paper; the minus charges turn the other way.

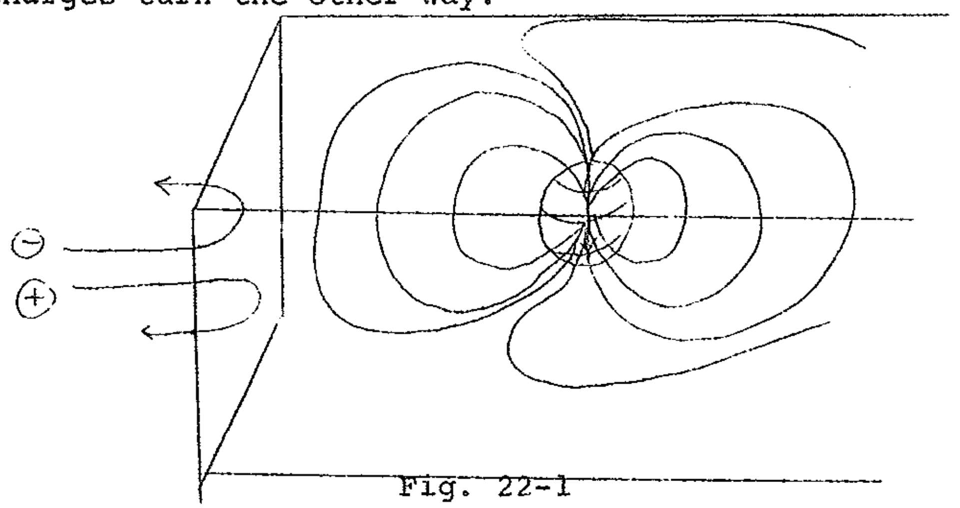
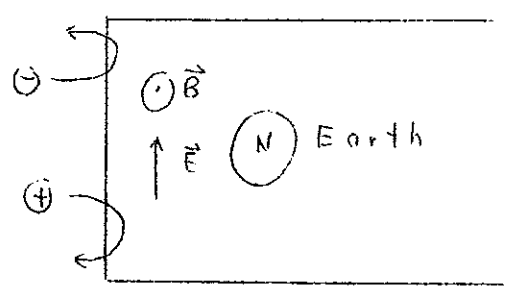


Fig. 22-1

Thus, there is a separation between plus and minus where the plus charges and the minus charges all shift a cylotron diameter in opposite directions. Now refer to Figure 22-2. If there were



View looking down on North Pole of Earth.

Fig. 22-2

no plasma in the earth's magnetic field, an electric field would be set up. The $\overline{E} \times \overline{B}$ drift from the E and B fields will cause the solar plasma to drift in to the plasma. Therefore, you can't really stop a streaming plasma unless you have an infinitely extended magnetic field. The classical formulation of the Chapman-Ferraro problem is thus incomplete; the application of it isn't really correct unless you have a plasma inside the magnetosphere. So, all planets have ionospheres supplied by the solar wind at least.

The particles of the ionosphere may not be permanent; they could go drifting right on thru. They will slow down as they come near the dipole, then they will drift around and go on out. However, if the drifting particles are stopped by the lower ionosphere, they will be stopped all the way out to the magnetopause (the outer boundary of the magnetosphere), because the electric field is shorted out by the plasma that is inside the magnetosphere. The whole inside of the protonosphere is an equipotential volume. There won't be a potential difference from one field line to the next in a steady state case. It means you can't have an electric field inside the plasma.

This is a subject of some controversy, I might add. Alfven, particularly, has theories where he gets electric fields outside penetrating thru this region. Most people don't buy that; they think that you can treat each magnetic field line as a conducting wire (the conductivity along the field line is very high).embedded in the ionosphere, which has good conductivity transverse to the magnetic field. Thus, you have a copper shell and copper wires coming out of it. Each one of these wires is an equipotential line. The transverse conductivity between wires at high altitudes is not very good because the collisions are quite infrequent. But down at the bottom, where there are a lot of collisions, you have the effect of a copper sphere. The copper sphere is an equipotential surface. As the whole thing is of equal potential, there can't be any electric field inside. I think that is a sound conclusion.

We are done for the present with the effect of the solar wind on the earth's magnetic field. Whether it closes into a teardrop shape as given in the Satellite Environment Handbook or whether it is blown open into a comet-like shape is a controversial point. Now we are going to talk about the effect of the spiral interplanetary magnetic field on cosmic radiation. After that we will talk about magnetic storms.

Let's go into interplanetary space and look at what happens to cosmic rays. Remember we have a spiral interplanetary magnetic field (the Archimedes spiral). Actually, the magnetic field is not always smooth; it has wiggles on it, so we are drawing an idealized average condition when we draw a nice, smooth spiral. But it is the way to get started. Then, if you want to make corrections, you put in fluctuations.

Let us define an angle χ as the angle between the normal to the magnetic field and the radius vector r. Previously, we talked about how, as the plasma moves radially out, in order to keep the plasma from cutting magnetic field lines, the magnetic field co-rotates with the sun as a rigid body. As the magnetic field turns, the plasma slides along the field line and moves radially out. The Archimedes spiral is arranged just so this works perfectly.

There will be an electric field out of the board because, if we have a moving magnetic field, the electric field $E = \Omega r B \sin\chi$.

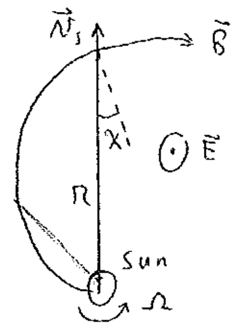


Fig. 22-3

David Cummings

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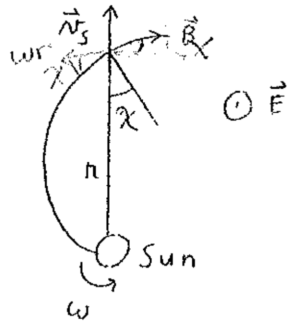
SPACE SCIENCE 500

Lecture XXIII

Last time we drew the spiral interplanetary field and said there was an electric field present in the region.



$$E = \omega r B \sin \chi$$

Fig. 23-1

The solar plasma doesn't see the electric field. It has a component of velocity perpendicular to B that is just equal to the drift velocity E/B. Suppose as a test particle, you put a stationary electron into the interplanetary medium. What would happen to it? Which way would it go? It would behave as if it were trapped or frozen onto the field line. There are two ways to look at the field. Either you can look at the field as moving radially out, or you could get the same answer by assuming the field co-rotates with the sun as a rigid body. The stationary test particle would move perpendicularly to B with a velocity E/B.

We should dwell on this for a few minutes because there are several ways of looking at it, and you want to be sure you could approach it from any point of view and get it right.

First of all, as seen from the earth, the plasma is moving and carrying the magnetic field with it. If the plasma is moving with a velocity V_s, the solar wind velocity $\overline{V}_s \times \overline{B}$ is the electric field.

If I reverse B, I would also reverse E. The $\overline{E} \times \overline{B}/B^2$ drift would be exactly the same. The plasma then has two motions; it has the motion parallel to B (and that motion doesn't count), and the motion perpendicular to B is just the drift velocity. In other words, the plasma is drifting in the E cross B direction as seen from the earth. In addition, it has the motion parallel to B so the net result is a motion radially out from the sun.

Now let's look at it from the frame of reference as seen by the solar wind. Let's make the field uniform and forget about the spiral for a minute. Thus, we have a uniform field tilted at an angle and a moving plasma as shown in Figure 23-2. From a frame of reference moving with the solar wind, there is no electric field. "Why not?" "What is going on?"

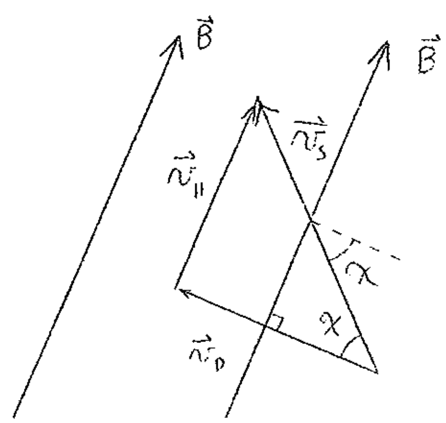


Fig. 23-2

We can resolve the velocity vector into components: a motion parallel and perpendicular to the magnetic field. Obviously, the parallel motion is not going to affect anything. Particles move parallel to the field with a simple spiral motion.

Then there is the motion perpendicular to B, but the way to look at it is that the magnetic field is frozen in and the magnetic field is moving with the plasma. Thus, the total velocity vector is broken up into a velocity parallel to B and a velocity perpendicular to B. So far as the solar wind is concerned, it sees the magnetic field moving perpendicular to B with a velocity that turns out to be V drift = $\frac{E}{B}$ as before. The important thing here is to be able to explain the motion with or without an electric field.

Let's look at the effect of the co-rotating interplanetary magnetic field on cosmic radiation. Remember that if I put in a test particle, it will not necessarily move with the solar wind. The solar wind is a very special plasma motion; it has the E/B drift velocity and it has an extra component of velocity parallel to the field. The guiding centers of cosmic rays do not necessarily have this parallel velocity component; in general, the cosmic rays just fill this region with an isotropic flux. There are two ways to look at it: (1) The guiding centers (if their cyclotron radius is small enough) will move and drift along with the velocity E/B for the guiding center, and so will the solar flare cosmic rays after they become isotropic. You don't want to forget about the E/B drift velocity just because it is small compared to c. There may be a subtle weak effect. In fact, the effect we are going to come up with is about .4 or .8% perturbation on an isotropic flux.

The second way to look at this is discussed in a subsequent lecture.

SPACE SCIENCE 500

Lecture XXIV

We will now examine the effect of the solar wind on cosmic radiation. An electric field is created in the interplanetary medium by the solar wind carrying the spiral solar magnetic field with it. Another way of looking at it is to say that the solar wind moves thru the spiral and causes the interplanetary magnetic field to co-rotate with the sun -- and both ways $\overline{V} \times \overline{B}$ turns out to give you the same electric field.

Now let's think of cosmic rays moving in across the electric field, E, and magnetic field, B. Here we have an electric field and a magnetic field -- magnetic field into the paper and electric field up.

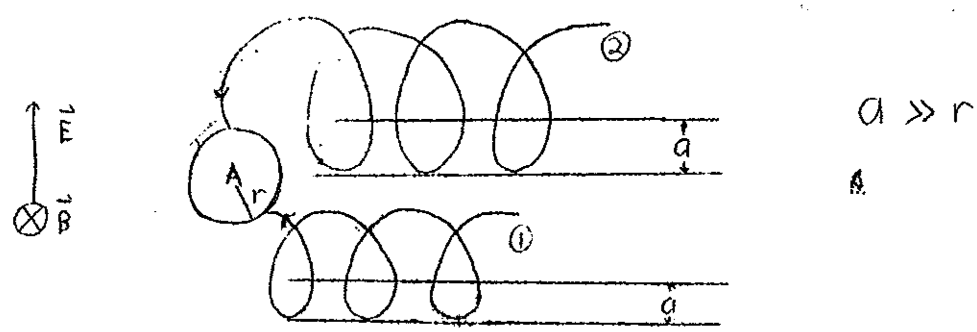


Fig. 24-1

Let's take a positive particle, ①. Let's put the earth right at A. The particle, as it strikes the earth from the right, will have an excess energy eEa. (We have chosen the reference for zero electrostatic energy as being along the guiding center.) As the particle moves up and down the electric field, it gains or loses energy.

Let's look at a second particle, (2). This particle comes looping along, and it can only strike the earth from the left side. The interesting thing about this particle is that it has lost energy by moving against the electric field, $\Delta\epsilon = eEa$.

Now there is another way of looking at this. We can move into the stationary frame of reference where we see no electric field. All you have to do is consider that you are moving with the guiding center.

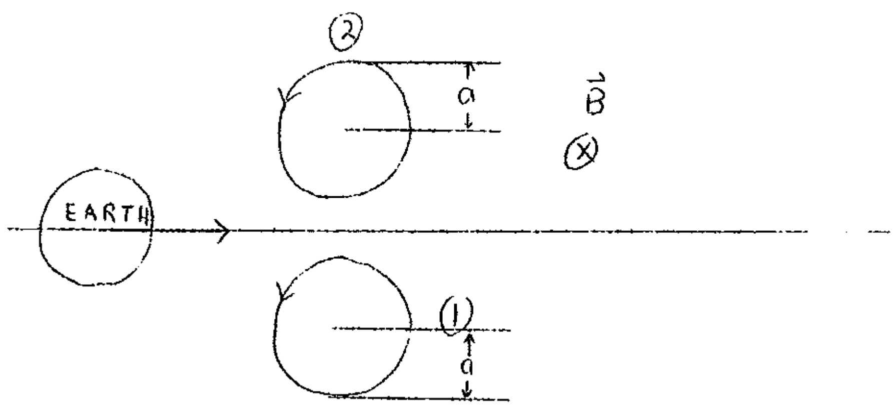


Fig. 24-2

Consider you are moving along with the guiding centers. You would see the earth coming along from the left and plowing into these cosmic rays as shown in Fig. 24-2. Just like a car driving thru rain and raindrops hit the front windshield harder than the back, the cosmic rays will hit harder on the forward side of the earth than on the back side. Thus, a definite anisotropy is produced perpendicular to the magnetic field. The magnitude of the cosmic ray diurnal variation is about one-half a percent.

There are three effects to be noted: (1) The energy spectr is shifted. Say the integral spectrum is as shown in Fig. 24-3.

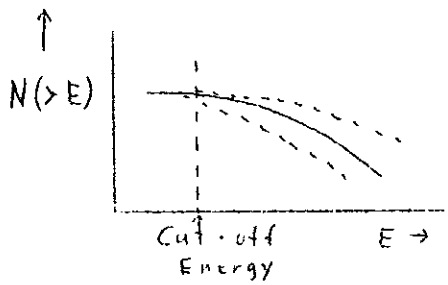


Fig. 24-3

At a given cosmic ray station, the cut-off energy is fixed. The area under the curve for a given energy cut-off goes up and down as the earth turns on its axis. (2) There is an aberration effect -- just like optical aberration, the same sort of thing that is done in astronomy. The same with raindrops again; they appear to come in a more and more parallel beam out in front, so the flux of particles per steradian solid angle becomes greater. And (3), the number of particles striking a given area per unit time goes up, like the number of raindrops striking the windshield per unit time per unit area goes up the faster and faster you drive. All three of these turn out to yield terms of the same order of magnitude, and they all add. As the earth turns in this system, then, you would expect to see a cosmic ray diurnal variation. And you do -- at just about the right order of magnitude. However, there are a lot of troubles in detail that still need to be worked out.

You ought to go thru these arguments and convince yourself that reversing the magnetic field, the interplanetary magnetic field, will not revérse the sign or the phase of the diurnal variation.

A bit of history here: At first people began to notice the cosmic ray diurnal variation with statistical analyses. Later, when they understood a little more about what was going on with the cosmic rays, they knew they were counting mainly mesons. The mesons were created at some altitude overhead, and then came down and struck the detector. But many mesons decayed in flight and never got to the detector. Thus, the intensity at the detector depended on the altitude of formation of the mesons. This altitude goes up and down diurnally, too; there are diurnal tidal effects in the atmosphere. The correction for this effect is about the same magnitude as the observed diurnal variation. Some said that the effect is all atmospheric -- but the phase was wrong. Finally Elliott, of Imperial College in England, did a very clever experiment. He put up some cosmic ray telescopes pointing at right angles to each other, each 45 degrees from the perpendicular. The atmosphere is really a very thin layer. Therefore

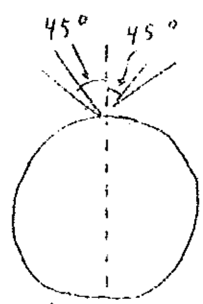


Fig. 24-4

both telescopes should see the same thing at the same time if there is no external anisotropy. However, if there is a real external anisotropy, then whatever one telescope sees, the other should see six hours later, if you are making the correct corrections for the change in atmospheric height of meson formation. And it is just the way it worked out; it really nailed down the conclusion that there was a real external anisotropy of the galactic cosmic rays that always occurred at about the same time of day. It was also established that this is not an anisotropy that is controlled by the galactic arm; it is controlled by the sun.

Let's work a little more on this and see what we get. We know the electric field is $V_D B$ and the cyclotron radius is mw/eB where w is the particle velocity. (We will do this classically, but we will discuss the relativistic results shortly.)

$$\Delta \text{ energy} = eEa$$
$$= e(V_D B)\left(\frac{mw}{eB}\right)$$
$$= mV_D w \quad .$$

That is the change in energy as the particles move up and down in the electric field. This is also the change in energy if we consider an isotropic flux in a stationary system (no electric field), and we get on the earth and begin moving thru this system with a velocity V_D:

$$\Delta \text{ energy} = \frac{1}{2}\, m(w + V_D)^2 - \frac{1}{2}\, mw^2 \quad .$$

The first term cancels, there is a crossproduct, and a second order term, but since w is much higher than the drift velocity, the second order term is negligible compared to $mV_D w$. Thus,

$$\Delta\varepsilon = \frac{1}{2}\, m\ (2wV_D + V_D{}^2)$$

$$\Delta\varepsilon = mwV_D \quad .$$

So, there are two ways of looking at this effect, either from the electric field or from the change of coordinate systems. You should be able to get the same description of nature in any coordinate system. You cannot use the argument -- and I keep bringing it up because it is used in this phenomenon -- the argument used is that since there is no electric field in one coordinate system, there isn't any electric field in any coordinate system. It may seem inconceivable to you that this is used, but it is.

HOMEWORK

Problem 1. Solve for Δ energy in two ways, both the change for a particle moving in an electric field and the change of energy of a particle when you change coordinate systems -- and do it relativistically. In other words, redo this lecture relativistically. See if you get a different result.

Problem 2. This problem will really clarify everything in a way that you wouldn't believe possible. You have a slab of plasma, infinite extent coming out of the paper, and a magnetic field B coming out of the paper inside the slab; no magnetic field outside. The whole slab of plasma is moving to the right with a uniform velocity V<<C.

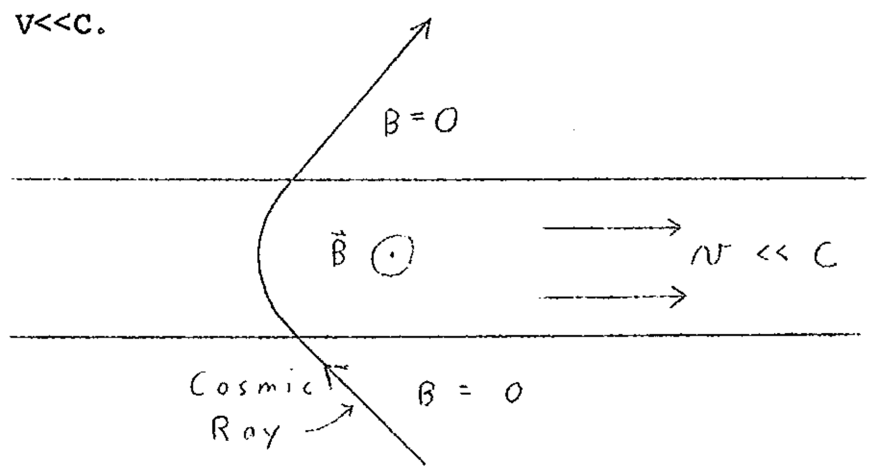


Fig. 24-5

If we are just standing and looking at this from the outside, we see an electric field. If we move with the plasma, we see no electric field. The question is: If in one frame of reference we see an electric field, and in the other frame of reference, we don't (one is a stationary frame and the other a moving frame), how can we get equivalent descriptions of cosmic ray particles coming thru because in one frame of reference the cosmic ray particle will either work against the electric field or have work done on it by the electric field. In the other frame of reference, there isn't any electric field. In other words, we go thru this and demonstrate that it doesn't matter which frame of reference is chosen.

DESSLER 12/16/63

SPACE SCIENCE 500

Lecture XXV

Let's take a uniformly magnetized sphere, spin it about its spin axis, and we will put a single electron in the magnetic field.

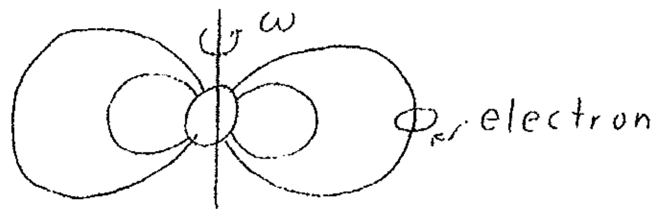


Fig. 25-1

What does the electron do? When a sphere is not spinning, the electron will drift in the gradient of the field. Now, let's start the sphere spinning. What happens? The way to look at this is that the magnetic field lines stand still and the sphere rotates inside them. The magnetic field lines do not co-rotate with the sphere. This point is covered in Cosmical Electrodynamics by Alfven, section 1.3, p. 6.

The sphere spins in its own field, and an electric quadrapole-like moment is generated as shown in Figure 25-2.

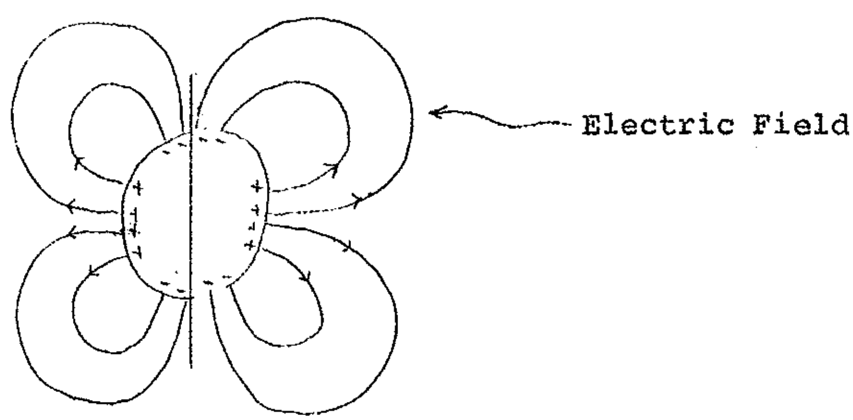


Fig. 25-2

The magnetic field from the sphere falls off as $1/r^3$ while the electric field falls off at least as fast as $1/r^4$. Thus, the E/B drift velocity decreases with radial distance so the particle will not in general have a drift velocity ωr that would be required if the magnetic field rotated with the magnet.

Now, let me build a new physical system. I will take two lead plates, and I will cut four big holes in them. There is an axis here about which I can spin one of the plates. I impress a magnetic field so it goes thru the holes and take the plate below the superconducting transition point so it becomes superconducting. A property of a superconductor is that no field line can penetrate the superconductor and the flux thru a hole is constant. Then I turn off the external field, and I am left with a field that looks like this.

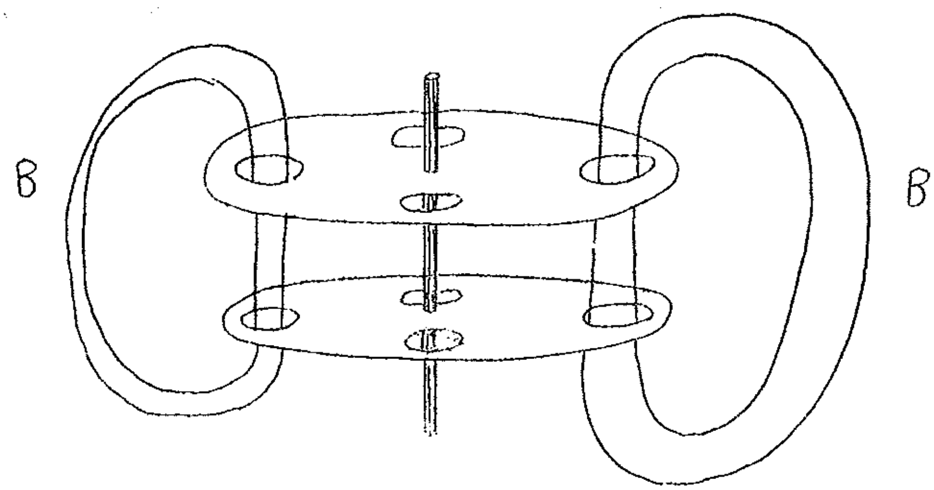


Fig. 25-3

I hold one plate rigidly, and I spin the other. Do I twist up the field lines like a rope in the middle? You think it would wind up? What do you base that on? All right, I'll make this center piece also superconducting. (I hadn't thought about that one.) The axis is as a superconducting tin rod. Before I have spun any plate, you could have solved for the field configuration (this would be a Physics 530 problem, for example). You are given regions where you know the total flux. You also know that the normal component of B is zero everyplace along the surface of the superconductor. Also, the only field we have is the field of the plate, so you know that far away the field falls off as $1/r^3$, and goes to zero. You have all the boundary conditions, and you can solve the field configuration in principle; you can solve it completely (although you might have to use a computer). Now when you get an answer, you get THE one and only answer. If you satisfy the boundary conditions in a static electromagnetic problem, then you have the unique answer. After you spin one plate N times, there still is only one answer. It depends upon the boundary conditions, and they haven't changed. Therefore, the magnetic field will not twist up like a rope if there is a vacuum between the plates. However, from the principle of frozen-in flux, we know the field would twist up. In this latter case, the boundary conditions would be altered by volume currents in the plasma.